Math 19. Mathematical Modeling
Stability in a Two-Component Linear System

Thomas W. Judson
Harvard University

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A two-component linear system is a system of differential equations of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by, & (1) \\ \frac{dy}{dt} &= cx + dy. \end{aligned}$$

Given initial conditions, $(x(0), y(0)) = (x_0, y_0)$, the system has a unique solution and is completely predictive. We can also write this system in matrix form as

$$\frac{d\mathbf{v}}{dt} = A\mathbf{v},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The initial conditions translate to

$$\mathbf{v}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

An ***equilibrium solution*** to the system is a solution where $\mathbf{v}(t) = (x(t), y(t))$ is a constant vector. Since the null clines of the system (1) intersect only at the origin, there is exactly one equilibrium point $\mathbf{0}$.

The constant solution $\mathbf{0}$ is said to be ***stable*** when *all* trajectories that start in some region with $\mathbf{0}$ inside move closer to $\mathbf{0}$ as $t \to \infty$. Otherwise, $\mathbf{0}$ is ***unstable***. The system $d\mathbf{v}/dt = A\mathbf{v}$ has a stable equilibrium point if and only if

$$\operatorname{tr}(A) = a + d < 0 \qquad (2)$$

$$\det(A) = ad - bc > 0. \qquad (3)$$

Unstable solutions are unlikely to be seen in nature.

To see how this works, we will examine the exact solutions to $d\mathbf{v}/dt = A\mathbf{v}$. There are two cases to consider. First suppose that $b = c = 0$. The system then becomes

$$\begin{aligned} \frac{dx}{dt} &= ax, \\ \frac{dy}{dt} &= dy. \end{aligned}$$

The solutions to this system are

$$\begin{aligned} x &= x_0 e^{at} \\ y &= y_0 e^{dt}. \end{aligned}$$

As $t \to \infty$, both of these solutions will tend towards zero exactly when both a and d are negative. This is equivalent to satisfying conditions (2) and (3).

The general case is a bit more complicated. We first introduce a vector

$$\mathbf{w}_0 = \begin{pmatrix} (a-d)x_0/2 + by_0 \\ cx_0 + (d-a)y_0/2 \end{pmatrix}.$$

We will also define Δ as

$$\Delta = \frac{1}{4}\operatorname{tr}(A)^2 - \det(A).$$

The precise form of the solution depends on Δ.

- If $\Delta = 0$, then the solution must be of the form

$$\mathbf{v}(t) = e^{\operatorname{tr}(A)t/2}(\mathbf{v_0} + t\mathbf{w}_0) \tag{4}$$

- If $\Delta > 0$, then the solution must be of the form

$$\mathbf{v}(t) = \frac{1}{2}e^{\operatorname{tr}(A)t/2}\left(e^{\sqrt{\Delta t}}(\mathbf{v_0} + \Delta^{-1/2}\mathbf{w}_0) + e^{-\sqrt{\Delta t}}(\mathbf{v_0} - \Delta^{-1/2}\mathbf{w}_0)\right) \tag{5}$$

- If $\Delta < 0$, then the solution must be of the form

$$\mathbf{v}(t) = \frac{1}{2}e^{\operatorname{tr}(A)t/2}\left(\cos(|\Delta|^{1/2}t)\mathbf{v_0} + |\Delta|^{-1/2}\sin(|\Delta|^{1/2}t)\mathbf{w}_0\right). \tag{6}$$

Deriving these solutions is a tough problem and would take several lectures; however, it is straightforward to check that (4), (5), and (6) are actually solutions. Simply differentiate and substitute into (1). Of course this is a little messy and tedious.

The question that we must answer is what happens to each of the solutions as $t \to \infty$. The solution corresponding to $\Delta = 0$ will go to zero exactly when $\operatorname{tr}(A) < 0$. You can apply l'Hôpital's Rule from calculus to take the limit. The same computation works for $\Delta < 0$, since sine and cosine are bounded by -1

and 1, but the case where $\Delta > 0$ is slightly more complicated. In this case, the solution (5) is the same as

$$\mathbf{v}(t) = \frac{1}{2}\left(e^{\mathrm{tr}(A)t/2+\sqrt{\Delta t}}(\mathbf{v_0} + \Delta^{-1/2}\mathbf{w}_0) + e^{\mathrm{tr}(A)t/2-\sqrt{\Delta t}}(\mathbf{v_0} - \Delta^{-1/2}\mathbf{w}_0)\right),$$

and we must examine the conditions that are necessary to guarantee that both $\mathrm{tr}(A)t/2 + \sqrt{\Delta t}$ and $\mathrm{tr}(A)t/2 - \sqrt{\Delta t}$ are negative (and hence the exponentials will go to zero as $t \to \infty$). However,

$$\begin{aligned}\frac{\mathrm{tr}(A)t}{2} + \sqrt{\Delta t} &= \frac{\mathrm{tr}(A)t}{2} + \sqrt{\left(\frac{\mathrm{tr}(A)^2}{4} - \det(A)\right)t} \\ &= \frac{\mathrm{tr}(A)t}{2} + \frac{\mathrm{tr}(A)}{2}\sqrt{\left(1 - \frac{4\det(A)}{\mathrm{tr}(A)}\right)t} \\ &= \frac{\mathrm{tr}(A)}{2}\left(t + \sqrt{\left(1 - \frac{4\det(A)}{\mathrm{tr}(A)}\right)t}\right),\end{aligned}$$

which negative if both (2) and (3) are satisfied, since $t > 0$. Similarly,

$$\frac{\mathrm{tr}(A)t}{2} - \sqrt{\Delta t} = \frac{\mathrm{tr}(A)}{2}\left(t - \sqrt{\left(1 - \frac{4\det(A)}{\mathrm{tr}(A)}\right)t}\right)$$

will be negative for large values of t. *WHEW!*