Math 19.

Name SOLUTIONS

Mathematical Modeling
Exam I—Fall 2003
T. Judson

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Problem Number	Possible Points	Score
1	8	
2	8	
3	5	
4	8	
5	6	
6	8	
7	10	
8	6	
9	8	
Total	67	

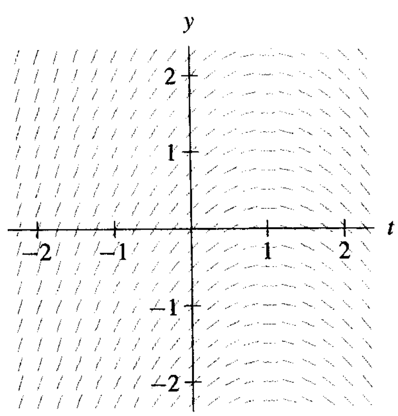
Directions—Please Read Carefully! You have two hours to take this midterm. Make sure to use correct mathematical notation. Any answer in decimal form must be accurate to three decimal places, unless otherwise specified. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You do not have to answer the problems in any particular order, so move to another problem if you find you are stuck or spending too much time on a single problem. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit (except if the directions for that question specifically say no justification is necessary, such as a True/False section). Please be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. You may use a calculator on this exam, but no other aids are allowed. **Good Luck!!!**

1. (8 points) Eight differential equations and four slope fields are given below. Determine the equation that corresponds to each slope field. No explanation is necessary.

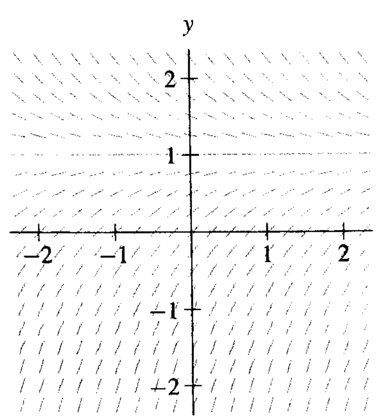
(i) $\dfrac{dy}{dt} = t - 1$	(v) $\dfrac{dy}{dt} = 1 - y$ (b)
(ii) $\dfrac{dy}{dt} = 1 - y^2$	(vi) $\dfrac{dy}{dt} = t^2 - y^2$
(iii) $\dfrac{dy}{dt} = y^2 - t^2$ (d)	(vii) $\dfrac{dy}{dt} = 1 + y$
(iv) $\dfrac{dy}{dt} = 1 - t$ (a)	(viii) $\dfrac{dy}{dt} = y^2 - 1$ (c)

(a)

(iv)



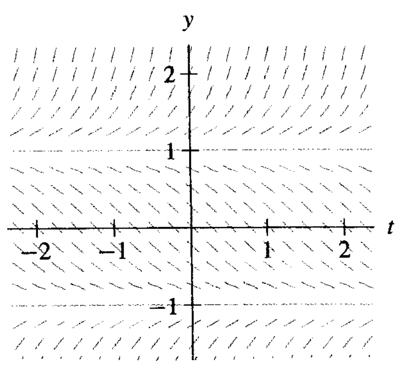
(b)



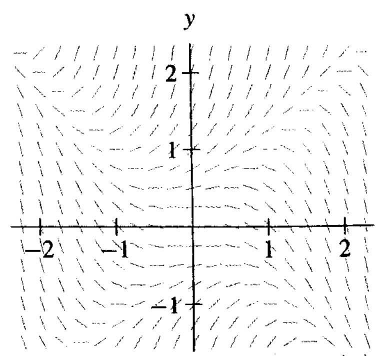
(v)

(c)

(viii)



(d)



(iii)

(a) Note that the slopes are constant along vertical lines—lines along which t is constant, so the right-hand side of the corresponding equation must depend only on t. The two such choices are equations (i) and (iv). Because the slope is negative for $t > 1$ and positive for $t < 1$, this slope field must correspond to equation (iv).

(b) This slope field has an equilibrium solution corresponding to the line $y = 1$, so it must be correspond to either equation (ii), (v), or (viii). Both (ii) and (viii) have another equilibrium solution corresponding to $y = -1$, so this slope field must correspond to equation (v).

(c) This slope field has equilibrium solutions corresponding to $y = \pm 1$. Hence it corresponds to either equation (ii) or (viii). Since dy/dt is negative along $y = 0$, this slope field must correspond to equation (viii).

(d) This slope field depends both on y and on t, so it can only correspond to equation (iii) or (vi). When $t = 0$ the slopes are positive, so the slope field must correspond to equation (iii).

2. (8 points) Consider the following differential equation for the function $y(t)$:

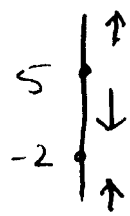
$$\frac{dy}{dt} = y^2 - 3y - 10.$$

(a) What are the equilibrium points?

$$\frac{dy}{dt} = y^2 - 3y - 10 = (y-5)(y+2)$$

$\Rightarrow$ EQUILIBRIUM POINTS ARE $y = 5$ AND $y = -2$

(b) Which equilibrium points are stable?



$\Rightarrow$ $y = -2$ is STABLE

(c) If $y(0) = 4$, what happens as t gets very large?

$y(t) \to -2$ AS $t \to \infty$

(d) If $y(0) = 6$, what happens as t gets very large?

$y(t) \to \infty$ AS $t \to \infty$

3. (5 points) The rhinoceros is now extremely rare. Suppose that enough game preserve land is set aside so that there is sufficient room for many more rhinoceros territories than there are rhinoceros. Consequently, there will be no danger of overcrowding. However, if the population is too small, fertile adults have difficulty finding each other when it is time to mate. Write a differential equation that models the rhinoceros population based on these assumptions. Write a *brief explanation* to support your model. (Note that there is more than one reasonable model that fits these assumptions).

Several different models are possible. Let $R(t)$ denote the rhinoceros population at time t. The basic assumption is that there is a minimum threshold that the population must exceed if it is to survive. In terms of the differential equation, this assumption means that dR/dt must be negative if R is close to zero. Three models that satisfy this assumption are:

- If k is a growth-rate parameter and M is a parameter measuring when the population is "too small", then
$$\frac{dR}{dt} = kR\left(\frac{R}{M} - 1\right).$$
- If k is a growth-rate parameter and b is a parameter that determines the level the population will start to decrease ($R < b/k$), then
$$\frac{dR}{dt} = kR - b.$$
- If k is a growth-rate parameter and b is a parameter that determines the extinction threshold, then
$$\frac{dR}{dt} = aR - \frac{b}{R}.$$

In each case, if R is below a certain threshold, dR/dt is negative. Thus, the rhinos will eventually die out. The choice of which model to use depends on other assumptions. There are other equations that are also consistent with the basic assumption.

4. Compute each of the following partial derivatives: (8 points)

(a) $\frac{\partial}{\partial x}(x^2y + 3x^2 + y^3)$

$= \boxed{2xy + 6x}$

(b) $\frac{\partial^2}{\partial x^2}(x^2y + 3x^2 + y^3)$

$= \boxed{2y + 6}$

(c) $\frac{\partial^2}{\partial x \partial y}(x^2y + 3x^2 + y^3)$

$= \boxed{2x}$

(d) $\frac{\partial}{\partial y}[x^2 \cos(x^2 + y^3)]$

$= -x^2 \sin(x^2 + y^3)[3y^2]$

$= \boxed{-3x^2y^2 \sin(x^2 + y^3)}$

5. (6 points) Consider the following two predator-prey systems of differential equations, where x represents the population of the prey and y represents the population of the predators.

(i)

$$\begin{aligned} \frac{dx}{dt} &= 5x - 3xy, \\ \frac{dy}{dt} &= -2y + \frac{1}{2}xy. \end{aligned}$$

(ii)

$$\begin{aligned} \frac{dx}{dt} &= x - 8xy, \\ \frac{dy}{dt} &= -2y + 6xy. \end{aligned}$$

(a) In which system does the prey reproduce more quickly when there are no predators?

(a) We consider dx/dt in each system. Setting $y = 0$ yields $dx/dt = 5x$ in system (i) and $dx/dt = x$ in system (ii). If the number x of prey is equal for both systems, dx/dt is larger in system (i). Therefore, the prey in system (i) reproduce faster if there are no predators.

(b) In which system are the predators most successful at catching prey? In other words, if the number of predators and prey are equal in both systems, in which system do the predators have a greater effect on the rate of change of the prey?

(b) We must see what affect the predators (represented by the y-terms) have on dx/dt in each system. Since the magnitude of the coefficient of the xy-term is larger in system (ii) than in system (i), y has a greater effect on dx/dt in system (ii). Hence the predators have a greater effect on the rate of change of the prey in system (ii).

(c) Which system requires more prey for the predators to achieve a given growth rate, assuming identical numbers of predators in both cases?

(c) We must see what affect the prey (represented by the x-terms) have on dy/dt in each system. Since x and y are both nonnegative, it follows that

$$-2y + \tfrac{1}{2}xy < -2y + 6xy,$$

and therefore, if the number of predators is equal for both systems, dy/dt is smaller in system (i). Hence more prey are required in system (i) than in system (ii) to achieve a certain growth rate.

6. (8 points) Consider the system of differential equations

$$\frac{dx}{dt} = 2x - y,$$
$$\frac{dy}{dt} = 3x + 2y.$$

(a) Draw and label the x and y null clines.

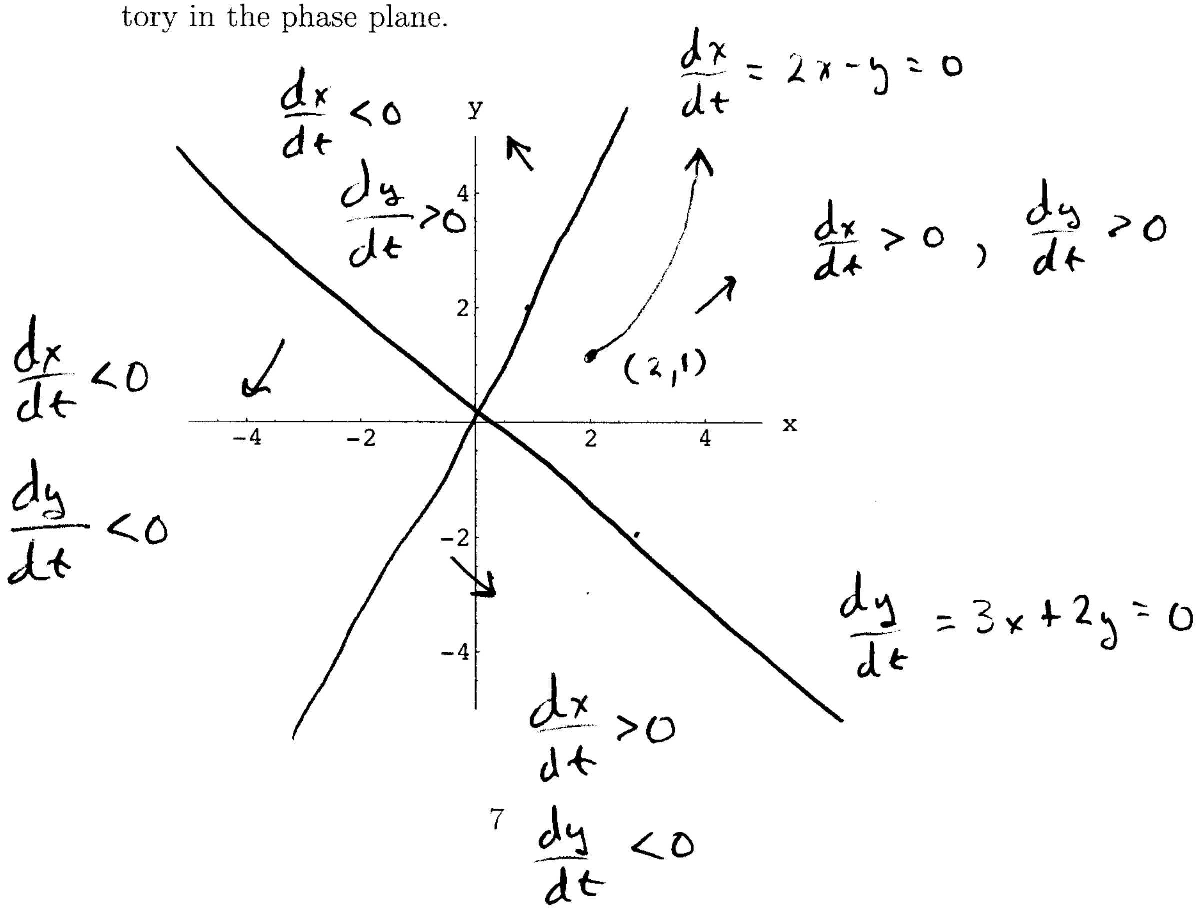
(b) Decide if $(0,0)$ is a stable equilibrium point. Justify your answer.

$$D = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$$

SINCE TR $(D) > 0$, THE ORIGIN IS $\boxed{\text{UNSTABLE.}}$

(c) On the drawing in part (a), label the regions where $dx/dt > 0$ and where $dx/dt < 0$. Do the same for dy/dt.

(d) For the initial condition $x(0) = 2$ and $y(0) = 1$, sketch the trajectory in the phase plane.

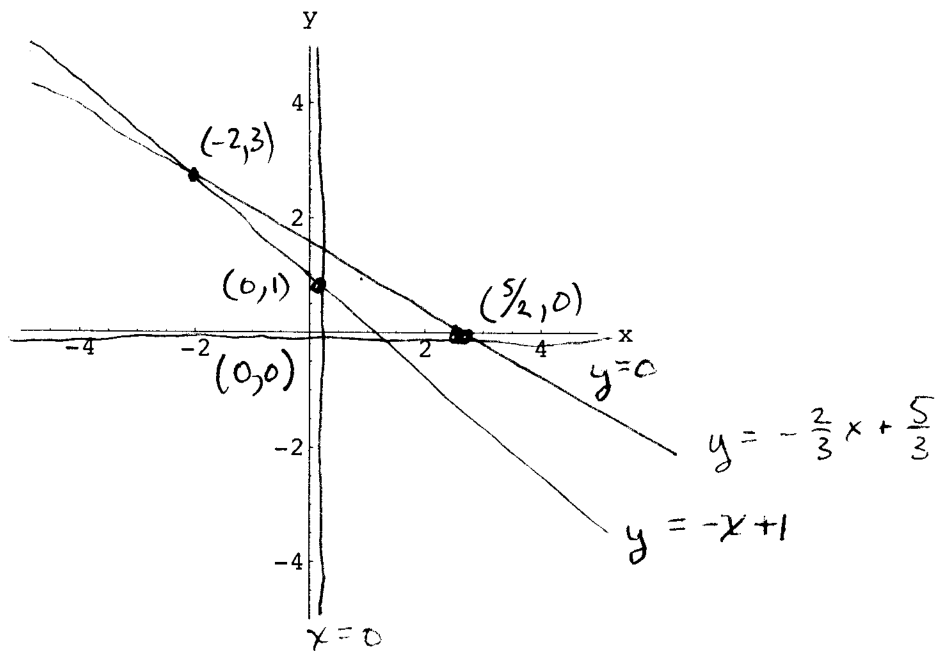


7. (10 points) Consider the system of differential equations

$$\frac{dx}{dt} = x(5 - 2x - 3y)$$
$$\frac{dy}{dt} = y(1 - x - y)$$

(a) Draw and label the x and y null clines.

(b) Find the equilibrium points and label them on your drawing from part (a).



X NULL CLINES

$x = 0$

$5 - 2x - 3y = 0$

or

$y = -\frac{2}{3}x + \frac{5}{3}$

y NULL CLINES

$y = 0$

$1 - x - y = 0$

or

$y = -x + 1$

$$
\begin{aligned}
\frac{dx}{dt} &= x(5-2x-3y) \\
\frac{dy}{dt} &= y(1-x-y)
\end{aligned}
$$

(c) Decide if the equilibrium points are stable. Justify your answer.

$$\frac{dx}{dt} = 5x - 2x^2 - 3xy$$

$$\frac{dy}{dt} = y - xy - y^2$$

$$D = \begin{bmatrix} 5 - 4x - 3y & -3x \\ -y & 1 - x - 2y \end{bmatrix}$$

$$D(0,1) = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$$ UNSTABLE

$$D(-2,3) = \begin{bmatrix} 4 & 6 \\ -3 & -3 \end{bmatrix}$$ UNSTABLE

$$D(5/2, 0) = \begin{bmatrix} -5 & -15/2 \\ 0 & -3/2 \end{bmatrix}$$ STABLE

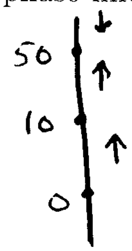
$$D(0,0) = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$ UNSTABLE

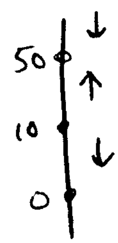
8. (6 points) Suppose that you wish to model a population with a differential equation of the form $dP/dt = f(P)$, where $P(t)$ is the population at time t. Experiments have been performed on the population that give the following information:

- The only equilibrium points in the population are $P = 0$, $P = 10$, and $P = 50$.
- If the population is 100, the population is decreasing.
- If the population is 25, the population is increasing.

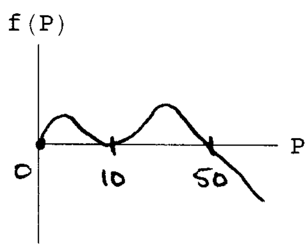
Answer each of the following questions.

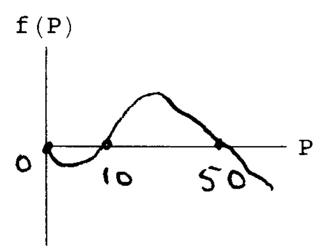
(a) Sketch the possible phase lines for this system for $P > 0$. (There are two.)





(b) Give a rough sketch of the corresponding functions $f(P)$ for each of your phase lines.





(c) Give a formula for the function $f(P)$ whose graph agrees (qualitatively) with the rough sketches in part (b) for each of your phase lines.

$f(P) = -P(P-10)^2(P-50)$

OR

$f(P) = -P(P-10)(P-50)$

9. (8 points) The following systems are models of the populations of a pair of species that either *compete* for resources (an increase in one species decreases the growth rate of the other species) or *cooperate* (an increase in one species increases the growth rate of the other species). For each system, identify the variables (independent and dependent) and the parameters (carrying capacity, measures of interaction between species, etc.) For each system say whether the species compete or cooperate. You may assume that all parameters are positive.

(a)

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \alpha \frac{x^2}{N} + \beta xy \\ \frac{dy}{dt} &= \gamma y + \delta xy \end{aligned}$$

(a) The independent variable is t, and x and y are dependent variables. Since each xy-term is positive, the presence of either species increases the rate of change of the other. Hence, these species cooperate. The parameter α is the growth-rate parameter for x, and γ is the growth-rate parameter for y. The parameter N represents the carrying capacity for x, but y has no carrying capacity. The parameter β measures the benefit to x of the interaction of the two species, and δ measures the benefit to y of the interaction.

(b)

$$\begin{aligned} \frac{dx}{dt} &= -\gamma x - \delta xy \\ \frac{dy}{dt} &= \alpha y - -\beta xy \end{aligned}$$

(b) The independent variable is t, and x and y are the dependent variables. Since both xy-terms are negative, these species compete. The parameter γ is the growth-rate coefficient for x, and α is the growth-rate parameter for y. Neither population has a carrying capacity. The parameter δ measures the harm to x caused by the interaction of the two species, and β measures the harm to y caused by the interaction.