Math 19. Lecture 20
Separation of Variables (II)

T. Judson

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1 Modeling the Density of Protein

It is known that the concentration of certain proteins at any cell in an embryo determines whether or not a particular gene is expressed in that cell. We will consider a cell model of an embryo where

$$u(t, x, y)$$

is the density of protein at time t and position (x, y). We will consider our embryo to be square, $[0, L] \times [0, L]$, where Protein is produced along the left-hand edge according to

$$u(t, 0, y) = \sin\left(\frac{\pi y}{L}\right).$$

Observe that this function is zero at $(0, 0)$ and $(0, L)$. Assume also that

$$\begin{aligned} u(t, x, 0) &= 0 \\ u(t, x, L) &= 0 \\ u(t, L, y) &= 0. \end{aligned}$$

The protein will diffuse according to the equation

$$\frac{\partial u}{\partial t} = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - ru.$$

Eventually, we will reach a steady-state

$$\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - ru = 0. \tag{1}$$

2 Separation of Variables

If

$$u(x, y) = A(x)B(y),$$

then equation (1) becomes

$$\mu(A''(x)B(y) + A(x)B''(y)) - A(x)B(y) = 0$$

or

$$\mu \left(\frac{A''(x)}{A(x)} + \frac{B''(y)}{B(y)} \right) = r.$$

The first term of the expression inside the parentheses of the last equation is a function of x and the second term is a function of y. Since x and y are independent variables and the equation is equal to a constant r, both of these terms must be constant. Therefore, we can assume that

$$\frac{1}{A} A'' = \frac{r}{\mu} - \lambda \tag{2}$$

$$\frac{1}{B} B'' = \lambda. \tag{3}$$

The boundary conditions now become

$$\begin{aligned} A(0)B(y) &= \sin\left(\frac{\pi y}{L}\right), \\ A(L)B(y) &= 0, \\ A(x)B(0) &= 0, \\ A(x)B(L) &= 0. \end{aligned}$$

We first solve $B'' = \lambda B$. There are three cases.

- If $\lambda > 0$, then
$$B = \alpha e^{\sqrt{\lambda} y} + \beta e^{-\sqrt{\lambda} y}.$$
- If $\lambda = 0$, then
$$B = \alpha + \beta y.$$
- If $\lambda < 0$, then
$$B = \alpha \cos \sqrt{|\lambda|}\, y + \beta \cos \sqrt{|\lambda|}\, y.$$

Applying the boundary condition $A(0)B(y) = \sin \pi/L$, the only consistent case occurs when $\lambda < 0$. If we let $\alpha = 0$ and $\beta = 1$, then

$$A(0)B(y) = \sin\left(\frac{\pi y}{L}\right),$$

and $\lambda = -\pi^2/L^2$.

Equation (2) now becomes

$$\frac{d^2 A}{dx^2} = \left(\frac{r}{\mu} + \frac{\pi^2}{L^2}\right) A.$$

To simplify matters, we will let

$$c = \frac{r}{\mu} + \frac{\pi^2}{L^2}.$$

Thus, we need to solve the equation

$$\frac{d^2 A}{dx^2} = cA.$$

In this case, $c > 0$. so the solutions must be of the form

$$A(x) = \alpha e^{\sqrt{c} x} + \beta e^{-\sqrt{c} x}.$$

Since

$$A(0)B(y) = \sin\left(\frac{\pi y}{L}\right),$$

$\alpha + \beta = 1$. Since $A(L)B(y) = 0$,

$$\alpha e^{\sqrt{c}L} + \beta e^{-\sqrt{c}L}.$$

Thus,

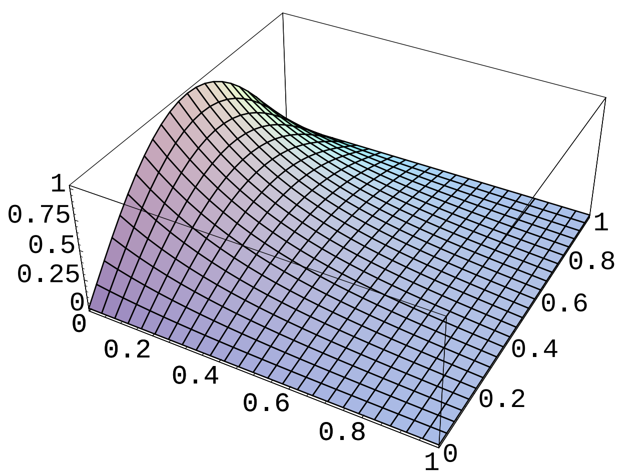
$$\begin{aligned} \alpha &= -\frac{1}{e^{2\sqrt{c}L} - 1} \\ \beta &= \frac{e^{2\sqrt{c}L}}{e^{2\sqrt{c}L} - 1} \end{aligned}$$

Thus,

$$u(x, y) = \frac{-e^{\sqrt{c}x} + e^{2\sqrt{c}L}e^{-\sqrt{c}x}}{e^{2\sqrt{c}L} - 1} \sin\left(\frac{\pi y}{L}\right)$$

where

$$c = \frac{r}{\mu} + \frac{\pi^2}{L^2}.$$



Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 17.