

Math 19. Lecture 32

Testing for Periodicity

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1 Testing for a Term with Period p

Certain phenomena may be the sum of periodic functions. For example,

$$m(t) = m_d(t) + m_m(t) + m_y(t)$$

might be a function depending on a daily cycle (m_d), and monthly cycle (m_m), and a yearly cycle (m_y). Yet, $m(t)$ itself may or may not look periodic.

- Suppose that we know $m(t)$ on the interval $[-\tau_0, -\tau_0 + \tau]$. Here τ_0 represents the amount of time before the present at which our data begins and τ is the time span for which we have collected data.
- We wish to know if our data $m(t)$ has a periodic component of period p . If p is much less than τ , we may be able to detect such a period.
- We must compute the following three integrals:

$$a \equiv \frac{1}{\tau} \int_{-\tau_0}^{-\tau_0+\tau} m(t) \cos(2\pi t/p) dt \quad (1)$$

$$b \equiv \frac{1}{\tau} \int_{-\tau_0}^{-\tau_0+\tau} m(t) \sin(2\pi t/p) dt \quad (2)$$

$$\sigma \equiv \frac{1}{\tau} \int_{-\tau_0}^{-\tau_0+\tau} m(t)^2 dt. \quad (3)$$

- With these numbers computed, we then compute

$$f = \frac{a^2 + b^2}{\sigma}, \quad (4)$$

which has a value between 0 and 1. A significant component of the original function $m(t)$ with period p is signified by a large value for f .

2 The Power Spectrum Function

In general, we may not have an a priori guess of what the periods of the components of m might be (if there are periods). In practice, a standard approach is to compute many values of f in (4) and see which values of p give us large f .

3 Fourier Coefficients

It is customary to change variables from the period p to the frequency $\nu = 1/p$. Then (1) and (2) become

$$a \equiv \frac{1}{\tau} \int_{-\tau_0}^{-\tau_0+\tau} m(t) \cos(2\pi\nu t) dt \quad (5)$$

$$b \equiv \frac{1}{\tau} \int_{-\tau_0}^{-\tau_0+\tau} m(t) \sin(2\pi\nu t) dt. \quad (6)$$

In this case, f becomes a function of ν :

$$f(\nu) = \frac{a(\nu)^2 + b(\nu)^2}{\sigma}.$$

This function is called the *power spectral density* function.

4 An Example

Suppose that

$$m(t) = \alpha \cos(2\pi t/q) + \beta \sin(2\pi t/q).$$

is periodic with period q . We compute the Fourier coefficients and the power spectral density function.

5 Trigonometric Integrals

Fourier coefficients can often be computed in exact form. We need the following indefinite integrals. If $A \neq B$, then

- $\int \cos(At) \cos(Bt) dt = \frac{\sin((A - B)t)}{2(A - B)} + \frac{\sin((A + B)t)}{2(A + B)}$
- $\int \sin(At) \cos(Bt) dt = \frac{\cos((A - B)t)}{2(A - B)} - \frac{\cos((A + B)t)}{2(A + B)}$
- $\int \sin(At) \sin(Bt) dt = \frac{\sin((A - B)t)}{2(A - B)} - \frac{\sin((A + B)t)}{2(A + B)}$

If $A = B$, then

- $\int \cos^2(At) dt = \frac{t}{2} + \frac{1}{4A} \sin(2At)$
- $\int \sin(At) \cos(At) dt = -\frac{1}{4A} \cos(2At)$
- $\int \sin^2(At) dt = \frac{t}{2} - \frac{1}{4A} \sin(2At)$

6 Example

Suppose that $m(t) = e^{-|t|}$ is defined on $(-\infty, \infty)$.

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$$\begin{aligned} a(\nu) &= \frac{1}{\tau} \int_{-\infty}^{\infty} e^{-|t|} \cos(2\pi\nu t) dt \\ &= \frac{2}{\tau} \int_0^{\infty} e^{-t} \cos(2\pi\nu t) dt \\ &= \frac{2}{\tau} \left[\frac{2\pi\nu e^{-t} \sin(2\pi\nu t) - e^{-t} \cos(2\pi\nu t)}{1 + 4\pi^2\nu^2} \right]_0^{\infty} \\ &= \frac{2}{\tau(1 + 4\pi^2\nu^2)} \end{aligned}$$

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$$b(\nu) = \frac{1}{\tau} \int_{-\infty}^{\infty} e^{-|t|} \sin(2\pi\nu t) dt = 0.$$

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$$\sigma = \frac{1}{\tau} \int_{-\infty}^{\infty} e^{-2|t|} dt = \frac{2}{\tau} \int_0^{\infty} e^{-2t} dt = \frac{2}{\tau} \left[-\frac{e^{-2t}}{2} \right]_0^{\infty} = \frac{1}{\tau}.$$

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$$f(\nu) = \frac{a(\nu)^2 + b(\nu)^2}{\sigma} = \frac{4}{\tau(1 + 4\pi^2\nu^2)^2}$$

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology*. Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 27.