Review Session 12/12/2004

– 2^{nd} Midterm

* See Review Sheet on the Course Website & the previous years' exams.
* Review some of the material from 1st half of the semester (i.e. determining nullclines)

(1) Advection Equation

$$\frac{\partial u}{\partial t} = -2\frac{\partial u}{\partial t} + 3u$$

$$u(0,x) = \frac{1}{1+e^x}$$

General form of Adv. eq.

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + r u$$

Solution : $u(t,x) = e^{-rt} f(x-ct)$

$$u(t,x) = e^{-3t} f(x-2t)$$

$$u(0,x) = \frac{1}{1+e^x} = f(x) \Rightarrow f(x-2t) = \frac{1}{1+e^{x-2t}} \Rightarrow$$

$$\Rightarrow \boxed{u(t,x) = e^{-3t}\cdot\frac{1}{1+e^{x-2t}}}$$

(2) Separation of Variables.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 3u$$

$u(t,-\pi) = u(t,\pi) = 0$ (boundary conditions)

$$u(t,x) = A(t)\cdot B(x)$$

Now, the diff. eq. becomes:

$$A'(t)\,B(x) = A(t)\,B'(x) - 3A(t)\,B(x)$$

Separate the var's!

$$\frac{A'(t)}{A(t)} = \frac{B''(x)}{B(x)} - 3 = \lambda$$

$A' = \lambda A \Rightarrow A(t) = A(t) \cdot e^{\lambda t}$

$A(t) \cdot B(-\pi) = A(t) \cdot B(\pi) = 0 \Rightarrow B(-\pi) = B(\pi) = 0$

$\Rightarrow B'' = (\lambda + 3) B = c B$

$c = 0 \Rightarrow B'' = 0 \Rightarrow B(x) = \alpha + \beta x$

$\left.\begin{array}{l} B(\pi) = \alpha + \beta\pi = 0 \\ B(-\pi) = \alpha - \beta\pi = 0 \end{array}\right\} \; 2\alpha = 0 \Rightarrow \alpha = 0 \Rightarrow$

$\Rightarrow \beta = 0$

Thus $\alpha = \beta = 0$ so no nontrivial solutions

$c > 0 \Rightarrow B(x) = \alpha e^{\sqrt{c}\,x} + \beta e^{-\sqrt{c}\,x}$

$\begin{cases} B(\pi) = \alpha e^{\sqrt{c}\pi} + \beta e^{-\sqrt{c}\pi} = 0 \;/\cdot e^{\sqrt{c}\pi} \\ B(-\pi) = \alpha e^{-\sqrt{c}\pi} + \beta e^{\sqrt{c}\pi} = 0 \;/\cdot e^{-\sqrt{c}\pi} \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} \alpha e^{2\sqrt{c}\pi} + \beta = 0 \\ \alpha e^{-2\sqrt{c}\pi} + \beta = 0 \end{cases} \Rightarrow \alpha\left(e^{2\sqrt{c}\pi} - e^{-2\sqrt{c}\pi}\right) = 0 \Rightarrow$

$\Rightarrow e^{2\sqrt{c}t} - e^{-2\sqrt{c}\pi} \neq 0 \Rightarrow \alpha = 0 \Rightarrow \beta = 0$

Thus $\alpha = \beta = 0$ so no nontrivial solutions.

$c < 0 \quad B(x) = \alpha \cos(\sqrt{-c}\,x) + \beta \sin(\sqrt{-c}\,x)$

$0 = B(\pi) = \alpha \cos(\sqrt{-c}\,\pi) + \beta \sin(\sqrt{-c}\,\pi)$

$0 = B(-\pi) = \alpha \cos(-\sqrt{-c}\,\pi) + \beta \sin(-\sqrt{-c}\,\pi) = \alpha \cos(\sqrt{-c}\,\pi) - \beta \sin(\sqrt{-c}\,\pi)$

Add the 2 expressions: $0 = 2\alpha \cos(\sqrt{-c}\,\pi)$

$(\sqrt{-c}\,\pi) = \frac{\pi}{2} + n\pi \Rightarrow \sqrt{-c} = \frac{1}{2} + n = \frac{2n+1}{2}$

Now $B(x) = \alpha \cos\left(\frac{2n+1}{2}x\right) + \beta \sin\left(\frac{2n+1}{2}x\right)$

Since $\sin\left(\frac{2n+1}{2}\pi\right) \neq 0 \Rightarrow \boxed{\beta = 0}$

So we get:

$$B(x) = \alpha \cdot \cos\left(\frac{2n+1}{2}x\right)$$

Check: $B(\pi) = \alpha \cos\left[\left(\frac{1}{2}+n\right)\pi\right] + \beta \sin\left[\left(\frac{1}{2}+n\right)\pi\right]$ $\left(\begin{matrix}\text{here}\\ \beta=0\end{matrix}\right)$

$\sqrt{-c} = \frac{1}{2}+n \Rightarrow \quad c = -\left(\frac{1}{2}+n\right)^2$

$\lambda + 3 = c = -\left(\frac{1}{2}+n\right)^2 \Rightarrow \lambda = -\left(\frac{1}{2}+n\right)^2 - 3;$

Thus the solution is:

$$u(t,x) = A(t)\cdot B(x) = e^{\left[-\left(\frac{1}{2}+n\right)^2 - 3\right]t} \cos\left[\left(\frac{1}{2}+n\right)x\right]$$

This solution is not blown up because the exponent of e is negative.

③ $f = \frac{dx}{dt} = x - 2y - (x^2+y^2)x$

$g = \frac{dy}{dt} = 2x + y - (x^2+y^2)y$

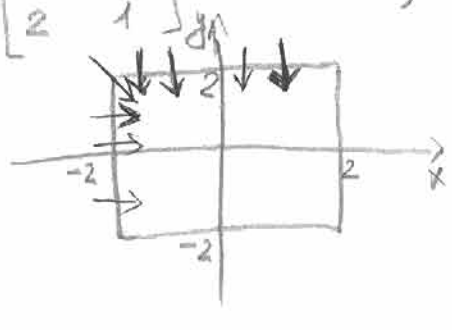
Want to show there is a periodic solution.

$(0,0)$ is an Equilibrium Solution.

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

$$J = \begin{bmatrix} 1-3x^2-y^2 & -2-2xy \\ 2-2xy & 1-x^2-3y^2 \end{bmatrix}$$

$J(0,0) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ tr, det $>0 \Rightarrow$ Repelling Equilibrium Solution.



Note: On the exam, there won't be a basin of attraction harder than a rectangle

$y=2, -2 \le x \le 2$

$$\frac{dy}{dt} = 2x + y - (x^2+y^2)y$$

$$\frac{dy}{dt} = 2x + 2 - 2(x^2+4)$$

$$\frac{dy}{dt} = -2x^2 + 2x - 6 \quad \text{(parabola open down)}$$

Diff. $-4x+2=0 \Rightarrow x=\frac{1}{2}$ is a critical point (where the maximum occurs)

$-2\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 6 = -\frac{1}{2} + 1 - 6 = -\frac{11}{2}$ Thus at the maximum it is neg. Thus it's always negative.

(4) $$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + 6\frac{\partial u}{\partial x} + u^2$$

$u(t,x) = f(x-ct)$ $\qquad$ $s = x - ct$

$$\frac{\partial u}{\partial t} = -c\, f'(x-ct) = -c\frac{df}{ds}$$

$$\frac{\partial u}{\partial x} = f'(x-ct) = \frac{df}{ds}$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x-ct) = \frac{d^2 f}{ds^2}$$

Substitute in the dif. equation: to get an ordinary eq (no partials)

$$-c\frac{df}{ds} = \mu\frac{d^2 f}{ds^2} + 6\frac{df}{ds} + f^2$$

$\boxed{\frac{df}{ds} = p} \Rightarrow \frac{d^2 f}{ds^2} = \frac{dp}{ds}$

We get 2 equations: a first-order system

$$-cp = \mu\frac{dp}{ds} + 6p + f^2$$

$$\frac{df}{ds} = p$$

$$\frac{dp}{ds} = \frac{-cp - 6p - f^2}{\mu}$$

(5) Principle of Superposition:

Any linear combination of 2 solutions is also a sol.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$u(t,0) = u(t,\pi) = 0$ Separate the variables to get these solutions.

$u_1(t,x) = e^{-t}\sin(x)$

$u_2(t,x) = e^{-4t}\sin(2x)$

$u_3(t,x) = e^{-9t}\sin(3x)$

Any linear combination of these solutions is also a solution.

Initial condition:

$u(0,x) = 60\sin(x) + 5\sin(2x) + 20\sin(3x)$

$u(t,x) = \alpha u_1 + \beta u_2 + \gamma u_3$; α, β, γ - ?

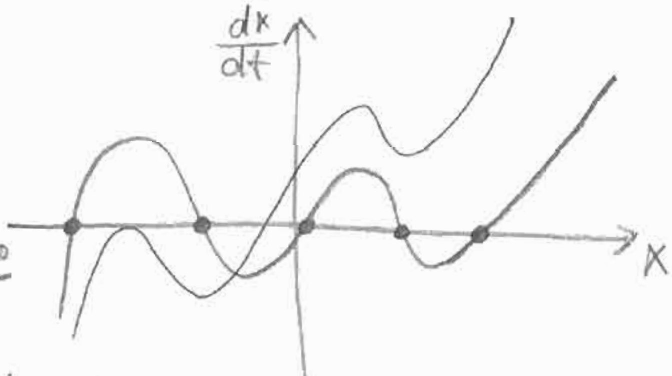
$u(0,x) = \alpha \sin x + \beta \sin(2x) + \gamma \sin(3x) \Rightarrow$ $\boxed{\alpha = 60 \quad \beta = 5 \quad \gamma = 20}$

$u(t,x) = 60u_1 + 5u_2 + 20u_3$

is the unique solution satisfying the initial condition.

(6) $\frac{dx}{dt} = f(x) + c$

$\frac{dx}{dt} = 3x^5 - 25x^3 + 60x + c$



Look at local max's and min's in order to be able to determine c

$f(x) = 3x^5 - 25x^3 + 60x + c$

$f'(x) = 15x^4 - 75x^2 + 60$

$f'(x) = 15(x^4 - 5x^2 + 4) = 15(x^2-4)(x^2-1) = 15(x-2)(x+2)(x-1)(x+1)$

Critical points at ± 2, ± 1

$$\frac{dx}{dt} = 3X^5 - 25x^3 + 60x + C$$

At $\boxed{x=1}$: $\frac{dx}{dt} = 3-25+60+C=0 \Rightarrow C=-38$

At $\boxed{x=2}$: $\frac{dx}{dt} = 3\cdot 32 - 25\cdot 8 + 60\cdot 2 + C = 0 \Rightarrow$

$\Rightarrow 96-200+120+c=0 \Rightarrow c=-16$

Do the other critical points the same way.

7 Stability

Check Linear Stability

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 5u(2-u)$$

$u_{e_1}(x) = 0$

$u_{e_2}(x) = 2$

$u(t,0) = u(t,1) = 0$

$f(u) = 10u - 5u^2$; $f'(u) = 10 - 10u$

$z(x) = f'(u_{e_1}(x)) = 10$

Find a pair (λ, g) with $\lambda > 0$ ang $g \not\equiv 0$ s.t.

$$\lambda g = \frac{d^2 g}{dx^2} - z(x) g \quad \left(\text{where } z(x) = f'(u_e(x))\right) \Rightarrow \text{then Solution - unstable}$$

$$\lambda g = \frac{d^2 g}{dx^2} - 10g$$

$$\frac{d^2 g}{dx^2} = (\lambda + 10) g$$

$\lambda + 10 = c$

$$\frac{d^2 g}{dx^2} = cg$$

If $c<0 \Rightarrow \lambda+10<0 \Rightarrow \lambda<-10$ impossible

$c=0 \Rightarrow \lambda=-10$ impossible

So $\boxed{c>0}$

$$g(x) = \alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x}$$

$0 = g(0) = \alpha + \beta = 0$; $\quad \beta = -\alpha$

$g(1) = 0 = \alpha e^{\sqrt{c}} + \beta e^{-\sqrt{c}} = \alpha\left(e^{\sqrt{c}} - e^{-\sqrt{c}}\right)$

Since $e^{\sqrt{c}} - e^{-\sqrt{c}} \neq 0 \Rightarrow \alpha = 0 \Rightarrow \alpha = \beta = 0 \Rightarrow g = 0$ everywhere

$\Rightarrow$ The solution is stable.

8 $\quad \lambda g = \frac{d^2 g}{dx^2} - (\cos x) g \qquad 0 \le x \le 1$

Use max principle to show that if $\lambda > 0$, $g = 0$

$$\frac{d^2 g}{dx^2} = \lambda g + (\cos x)\cdot g = (\lambda + \cos x) g$$

Can it have a pos. max and a neg. min, it has to be identically 0. (which is what we wanted to show)