Math 19. Lecture 9
Equilibrium in Two Component Systems

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1 Uniqueness of Solutions

A two-component linear system is a system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= ax + by, \\ \frac{dy}{dt} &= cx + dy. \end{aligned}$$

Given initial conditions, $(x(0), y(0)) = (x_0, y_0)$, the system has a unique solution and is completely predictive. We can also write this system in matrix form as

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \text{ and } \mathbf{x}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}.$$

An *equilibrium solution* to the system where $\mathbf{x}(t) = (x(t), y(t))$ is a constant vector.

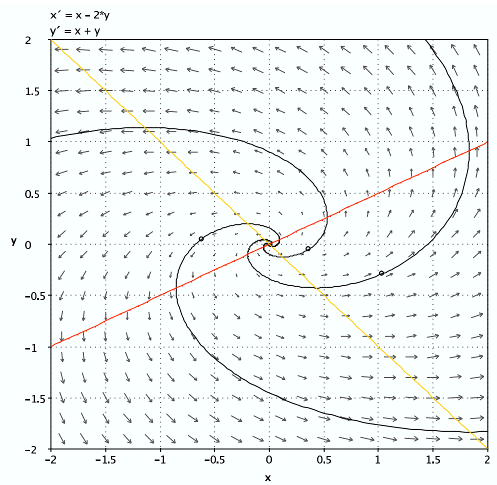
2 Determinants

The system $\mathbf{x}'(t) = A\mathbf{x}(t)$ has an equilibrium solution at $(0, 0)$ if it has only the solution $x = y = 0$. Another way of viewing this fact is to observe that the two lines

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

are not parallel if and only if $ad - bc \neq 0$. We define the *determinant* of A to be

$$\det(A) = ad - bc.$$



3 Stability Criterion

The constant solution $\mathbf{0}$ is said to be *stable* when *all* trajectories that start in some region with $\mathbf{0}$ inside move closer to $\mathbf{0}$ as $t \to \infty$. Otherwise, $\mathbf{0}$ is *unstable*. The system $\mathbf{x}' = A\mathbf{x}$ is stable if and only if

$$\begin{aligned} \mathrm{tr}(A) &< 0 \\ \det(A) &> 0. \end{aligned}$$

4 An Equation for $x(t)$

Let us examine 2×2 linear systems more closely. Let

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

where $x(0) = x_0$ and $y(0) = y_0$.

4.1 An Uncoupled System

Let us first assume that $b = c = 0$. Then the solution to the system

$$\begin{aligned} x' &= ax \\ y' &= dy \end{aligned}$$

is

$$\begin{aligned} x &= x_0 e^{at} \\ y &= y_0 e^{dt}. \end{aligned}$$

This system is stable if both a and d are negative. This occurs exactly when $\det(A) > 0$ and $\mathrm{tr}(A) < 0$.

4.2 The General Case

For the general case, we will let

$$\mathbf{x}_0 = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \text{ and } \mathbf{w}_0 = \begin{pmatrix} (a-d)x(0)/2 + by(0) \\ cx(0) + (d-a)y(0)/2 \end{pmatrix}$$

and

$$\Delta = \frac{1}{4}\,\mathrm{tr}(A)^2 - \det(A).$$

We have exactly three types of solutions.[1]

- *Case 1*: $\Delta > 0$.
$$\mathbf{x}(t) = \frac{1}{2} e^{\mathrm{tr}(A)t/2} \left(e^{\sqrt{\Delta t}}(\mathbf{x}_0 + \Delta^{-1/2}\mathbf{w}_0) + e^{-\sqrt{\Delta t}}(\mathbf{x}_0 - \Delta^{-1/2}\mathbf{w}_0) \right).$$

- *Case 2*: $\Delta = 0$.
$$\mathbf{x}(t) = e^{\mathrm{tr}(A)t/2}(\mathbf{x}_0 + t\mathbf{w}_0)).$$

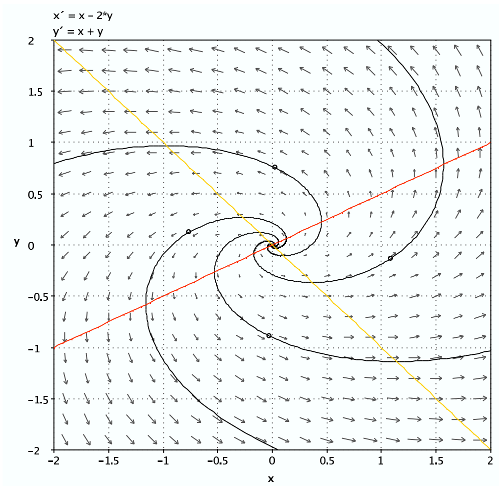
- *Case 3*: $\Delta < 0$.
$$\mathbf{x}(t) = \frac{1}{2} e^{\mathrm{tr}(A)t/2} \left(\cos\left(|\Delta|^{1/2} t\right) \mathbf{x}_0 + |\Delta|^{-1/2} \sin\left(|\Delta|^{1/2} t\right) \mathbf{w}_0 \right).$$

[1] These solutions can be derived using linear algebra. See Math 21b or Math 106.

In each case, you can get an unstable solution if $\mathbf{x}_0$ is chosen poorly and the conditions

$$\begin{aligned} \mathrm{tr}(A) &< 0 \\ \det(A) &> 0 \end{aligned}$$

are violated.



Homework

- Chapter 8. Exercises 1, 2, 3, 4, 5, 7; pp. 138–139.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 8.
- "Better Protection for the Ozone Layer," pp. 131–138.