Math 19. Lecture 28
Periodic Solutions

T. Judson

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1 An Improved Predator-Prey Model

If $p(t)$ be the number of prey and $q(t)$ is the number of predators at time t, then we can model a predator-prey system as

$$\frac{dp}{dt} = \frac{2}{3}p\left(1 - \frac{p}{4}\right) - \frac{pq}{1+p} \tag{1}$$

$$\frac{dq}{dt} = sq\left(1 - \frac{q}{p}\right), \tag{2}$$

where $s > 0$.

- If there are no predators, our system (1) is just logistic growth:

  $$\frac{dp}{dt} = \frac{2}{3}p\left(1 - \frac{p}{4}\right).$$

  We have a stable equilibrium at $p = 4$.

- The existence of predators decreases dp/dt by $pq/(1+q)$.

  – When p is small,

  $$\frac{pq}{1+p} \approx pq.$$

  This tells us that the dp/dt is dependent on predator-prey interaction.

– When p is large,

$$\frac{pq}{1+p} \approx q.$$

In other words, food is abundant and the death rate is only dependent on the number of predators.

- In (2), we have the standard logistic equation if p is constant:

$$\frac{dq}{dt} = sq\left(1 - \frac{q}{p}\right).$$

A lion can only eat so much! Thus, this equation models the fact that the carrying capacity for the predator is proportional to the number of prey.

2 The Phase Plane

- The p null clines are $p = 0$ and $q = (2/3)(1 - p/4)(1 + p)$.
- The q null clines are $q = 0$ and $q = p$.
- The equilibrium points for $p > 0$ are

$$\begin{aligned} (p,q) &= (1,1) \\ (p,q) &= (4,0). \end{aligned}$$

3 Stability

- At $p = 1$, $q = 1$,

$$A = \begin{pmatrix} 1/12 & -1/2 \\ s & -s \end{pmatrix}.$$

In this case,

$$\begin{aligned} \mathrm{tr}(A) &= \frac{1}{12} - s \\ \det(A) &= \frac{5}{12}s. \end{aligned}$$

This point is a stable equilibrium point if $s > 1/12$ and unstable if $s < 1/12$.

- At $p = 4$, $q = 0$,

$$A = \begin{pmatrix} -2/3 & -4/5 \\ 0 & s \end{pmatrix}.$$

In this case, $\det A = -2s/3 < 0$. Therefore, this point is not stable.

4 A Repelling Equilibrium Point

An equilibrium point is a *repelling equilibrium point* if whenever a non-equilibrium solution is close to the equilibrium solution at t, it moves further away as t increases. The equilibrium point $p = 1$ and $q = 1$ is repelling if $s < 1/12$. If $p(t)$ and $q(t)$ are both near 1, then

$$\begin{pmatrix} p \\ q \end{pmatrix}$$

is almost a solution to

$$\frac{d}{dt}\begin{pmatrix} p-1 \\ q-1 \end{pmatrix} = \begin{pmatrix} 1/12 & -1/2 \\ s & -s \end{pmatrix}\begin{pmatrix} p-1 \\ q-1 \end{pmatrix} = \begin{pmatrix} (p-1)/12 - (q-1)/12 \\ s(p-1) - s(q-1) \end{pmatrix}.$$

These solutions grow exponentially with time. An equilibrium point is repelling if $\mathrm{tr}(A) > 0$ and $\det(A) > 0$.

5 Basin of Attraction

A *basin of attraction* or a *trapping region* is a region V in the (p, q)-plane where no solution

$$\begin{pmatrix} p(t) \\ q(t) \end{pmatrix}$$

of our predator-system that enters V ever leaves V. We claim that the square region

$$V = \{(p, q) : 0 < p < 4, 0 < q < 4\}$$

is a basin of attraction.

6 Poincaré-Bendixson Theorem

Consider the system

$$\begin{aligned}\frac{dp}{dt} &= f(p,q)\\ \frac{dq}{dt} &= g(p,q),\end{aligned}$$

and suppose that a region V is a basin of attraction in the (p,q)-plane. If V contains a single equilibrium point that is repelling, then the system has a periodic solution that is inside V for all t.

7 Periodic Solutions

By the Poincaré-Bendixson Theorem, our predator-prey system has a periodic solution if $s < 1/12$.

8 Stability

Our periodic solution is stable in the following sense. Starting inside the periodic solution, a trajectory will spiral out towards the stable orbit. Starting outside the periodic solution, a trajectory will spiral in towards the stable orbit.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 23.
- "Snowshoe Hare Populations: Squeezed from Below and Above," pp. 382–385
- "Impact of Food and Predation on the Snowshoe Hare Cycle," pp. 385–391.