Math 19. Lecture 10
Stability in Nonlinear Systems

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1 Another Epidemic Model

Let us consider a more complicated epidemic model, say for a city. We make the following assumptions.

- Susceptibles enter the city at a constant rate of β persons per day.
- Infected people recover or die after a certain number of days. If they recover, they are immune.

If

$$\begin{aligned} S(t) &= \text{the number susceptible at time } t, \\ I(t) &= \text{the number infected at time } t, \end{aligned}$$

then we have the following system of differential equations:

$$\begin{aligned} \frac{dS}{dt} &= -\alpha SI + \beta \\ \frac{dI}{dt} &= -\sigma I + \alpha SI. \end{aligned}$$

Here, α is a constant that is determined by the probability of an interaction between a susceptible and an infected, and $1/\sigma$ is the mean time that a person is infectious. The model has a single equilibrium point at

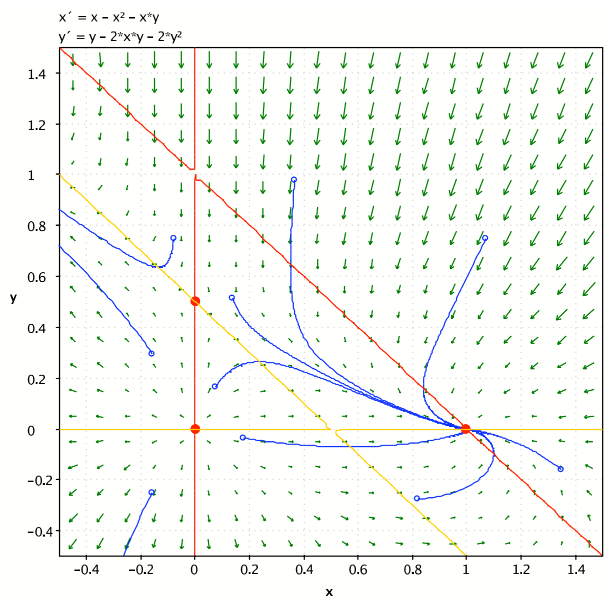
$$\begin{aligned} S &= \frac{\sigma}{\alpha} \\ I &= \frac{\beta}{\sigma}. \end{aligned}$$

We are interested in the stability of this equilibrium point. The answer is more complicated than before since our equation is nonlinear.

2 A Concrete Example

Suppose that we are given the system

$$\begin{aligned} x' &= x - x^2 - xy \\ y' &= y - 2xy - 2y^2. \end{aligned}$$



3 The General Case

If we have concrete numbers to work with, we can use software to find stable and unstable equilibrium points, but what about our epidemic example? To determine the general case, we must approximate a nonlinear system

$$\begin{aligned} x'(t) &= f(x, y) \\ y'(t) &= g(x, y) \end{aligned}$$

near each equilibrium point (x_0, y_0) with a linear system. To do this, we will use a Taylor series in two variables.

4 Partial Derivatives

A *partial derivative* for a function $f(x,y)$ is the derivative with respect to one of the variables. For example,

$$\frac{\partial}{\partial x} f(x,y)$$

tells us how f changes in the x direction. We define

$$\frac{\partial}{\partial x} f(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

We define

$$\frac{\partial}{\partial y} f(x,y)$$

similarly.

5 Taylor Series

The idea is to expand $f(x,y)$ and $g(x,y)$ into a Taylor series and then use the fact that any smooth function is locally linear and apply the methods of the previous lecture. For example,

$$f(x,y) = f(x_0,y_0) + \frac{\partial}{\partial x} f(x_0,y_0)(x - x_0) + \frac{\partial}{\partial y} f(x_0,y_0)(y - y_0) + \cdots \quad (1)$$

near the point (x_0, y_0).

6 Criterion for Stability

Let $\mathbf{x_0} = (x_0, y_0)$ and define

$$a = \left.\frac{\partial f}{\partial x}\right|_{\mathbf{x}_0}, b = \left.\frac{\partial f}{\partial x}\right|_{\mathbf{x}_0}, c = \left.\frac{\partial g}{\partial x}\right|_{\mathbf{x}_0}, d = \left.\frac{\partial f}{\partial x}\right|_{\mathbf{x}_0}.$$

If we define

$$D = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then $\mathbf{x}_0$ is a stable equilibrium point if

$$\begin{aligned} \det D &> 0 \\ \operatorname{tr} D &< 0. \end{aligned}$$

7 The Chain Rule

The two variable chain rule can be useful in these computations. Recall the single variable chain rule

$$\frac{d}{dt}f(g(t)) = f'(g(t))g'(t).$$

The two variable chain rule is

$$\frac{d}{dt}h(x(t), y(t)) = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt}.$$

Homework

- Chapter 9. Exercises 1, 2, 4, 5; pp. 145–146.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology*. Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 9.
- P. Blanchard, R. Devaney, and G. Hall. *Differential Equations*, second edition. Brooks/Cole, Pacific Grove, CA, 2002, pp. 444-446.