Math 19. Lecture 24
Stability Criterion (II)

T. Judson

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1 Linear Stability Criterion

Let u_e be an equilibrium solution to

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + f(u) \tag{1}$$

$$\frac{\partial}{\partial x} u(t,0) = \frac{\partial}{\partial x} u(t,L) = 0. \tag{2}$$

The solution $u_e(x)$ is a stable solution to

$$\mu \frac{d^2 u_e}{dx^2} + f(u_e) = 0 \tag{3}$$

$$\frac{d}{dx} u_e(0) = \frac{d}{dx} u_e(L) = 0 \tag{4}$$

if and only if there is *no* pair (g, λ) , where $g(x)$ is some function that is *not* identically zero for $0 \leq x \leq L$, where $\lambda \in \mathbb{R}$, and where the following constraints are satisfied.

- $\lambda \geq 0$
- $\lambda g = \mu \frac{d^2}{dx^2} + z(x) g$
- $\left.\frac{dg}{dx}\right|_{x=0} = \left.\frac{dg}{dx}\right|_{x=L} = 0$

A solution is unstable if there is even one such pair (g, λ) that obeys the above conditions.

2 Important Remarks about Stability

- For a specific u_e and $f(u)$ (hence $z(x)$), we may or may not be able to find such g and λ.

- If w is a solution to

$$\frac{\partial w}{\partial t} = \mu \frac{\partial^2 w}{\partial x^2} + z(x)w \tag{5}$$

$$\frac{\partial}{\partial x} w(t, 0) = \frac{\partial}{\partial x} w(t, L) = 0. \tag{6}$$

  and if $|w|$ is very small at all points x, then the function of space and time, $u_e(x) + w(t, x)$, is an approximate solution to (1) and (2).

- Conversely, if $u(t, x) = u_e(x) + w(t, x)$ is a solution to (1) and (2) for small $|w|$, then w will be an approximate solution for (5) and (6).

- If w solution to (5) and (6) such that $|w|$ is very small to begin with for all x but grows as $t \to \infty$ for some x, then (1) and (2) will have a solution that is close to $u_e(x)$ to start with but departs from $u_e(x)$ as $t \to \infty$. This solution can be approximated by $u(t, x) = u_e(x) + w(t, x)$.

  Conversely, if (1) and (2) have a solution of the form $u(t, x) = u_e(x) + w(t, x)$ that starts at $t = 0$ for small $|w|$ at all x, then (5) and (6) will have a solution that starts small and grows with time. This solution can be approximated by w when t is small.

- If all solutions w to (5) and (6) shrink in absolute value as $t \to \infty$, then all solutions to (1) and (2) that start near enough to the equilibrium solution $u_e(x)$ at $t = 0$ will approach $u_e(x)$ at all x as $t \to \infty$.

3 Boundary Conditions

The boundary condtions must match. Let u_e be an equilibrium solution to

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + f(u) \tag{7}$$

$$u(t, 0) = u(t, L) = 0. \tag{8}$$

The solution $u_e(x)$ is a stable solution to

$$\mu \frac{d^2 u_e}{dx^2} + f(u_e) = 0$$
$$u_e(0) = u_e(L) = 0$$

if and only if there is *no* pair (g, λ) , where $g(x)$ is some function that is *not* identically zero for $0 \le x \le L$, where $\lambda \in \mathbb{R}$, and where the following constraints are satisfied.

- $\lambda \ge 0$
- $\lambda g = \mu \frac{d^2}{dx^2} + z(x) g$
- $\left.\frac{dg}{dx}\right|_{x=0} = \left.\frac{dg}{dx}\right|_{x=L} = 0$

A solution is unstable if there is even one such pair (g, λ) that obeys the above conditions.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 19.
- "Direct and Continuous Assessment by Cells of Their Position in a Morphogen Gradient," pp. 296–300.
- "Activin Signalling and Response to a Morphogen Gradient," pp. 300–309.