Math 19. Name $\boxed{\text{SOLUTIONS}}$

Mathematical Modeling
Exam I—Fall 2004
T. Judson

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Problem Number	Possible Points	Score
1	25	
2	6	
3	8	
4	10	
5	10	
6	10	
7	12	
8	11	
9	8	
Total	100	

Directions—Please Read Carefully! You have two hours to take this midterm. Make sure to use correct mathematical notation. Any answer in decimal form must be accurate to three decimal places, unless otherwise specified. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You do not have to answer the problems in any particular order, so move to another problem if you find you are stuck or spending too much time on a single problem. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit (except if the directions for that question specifically say no justification is necessary, such as a True/False section). Please be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. You may use a calculator on this exam, but no other aids are allowed. **Good Luck!!!**

1. (25 points) In an isolated region of the Canadian Northwest Territories, a population of arctic wolves, $x(t)$, and a population of silver foxes, $y(t)$, compete for survival. (For each population, one unit represents 100 individuals). The two species have a common, limited food supply, which consists mainly of mice. The interaction of the two species can be modeled by the following system of differential equations,

$$\begin{aligned} \frac{dx}{dt} &= x - x^2 - xy \\ \frac{dy}{dt} &= \frac{3}{4}y - y^2 - \frac{1}{2}xy, \end{aligned}$$

where the proportionality constants were obtained from observation.

(a) Find the nullclines of the system for $x \geq 0$ and $y \geq 0$.

x-nullclines

$x = 0$

$y = -x + 1$

y-nullclines

$y = 0$

$y = -\frac{1}{2}x + \frac{3}{4}$

(b) Find all of the equilibrium solutions for $x \geq 0$ and $y \geq 0$.

$(0, 0)$

$(0, 3/4)$

$(1, 0)$

$(1/2, 1/2)$

(c) Using linearization, determine the nature of the equilibrium solution lies strictly in the first quadrant. That is, determine the stability of the equilibrium solution for $x > 0$ and $y > 0$. What is the long term situation for the foxes and the wolves? Can the two species survive together?

$$D = \begin{bmatrix} \frac{\partial}{\partial x}\left(x - x^2 - xy\right) & \frac{\partial}{\partial y}\left(x - x^2 - xy\right) \\ \frac{\partial}{\partial x}\left(\frac{3}{4}y - y^2 - \frac{1}{2}xy\right) & \frac{\partial}{\partial y}\left(\frac{3}{4}y - y^2 - \frac{1}{2}xy\right) \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 2x - y & -x \\ -\frac{1}{2}y & \frac{3}{4} - 2y - \frac{1}{2}x \end{bmatrix}$$

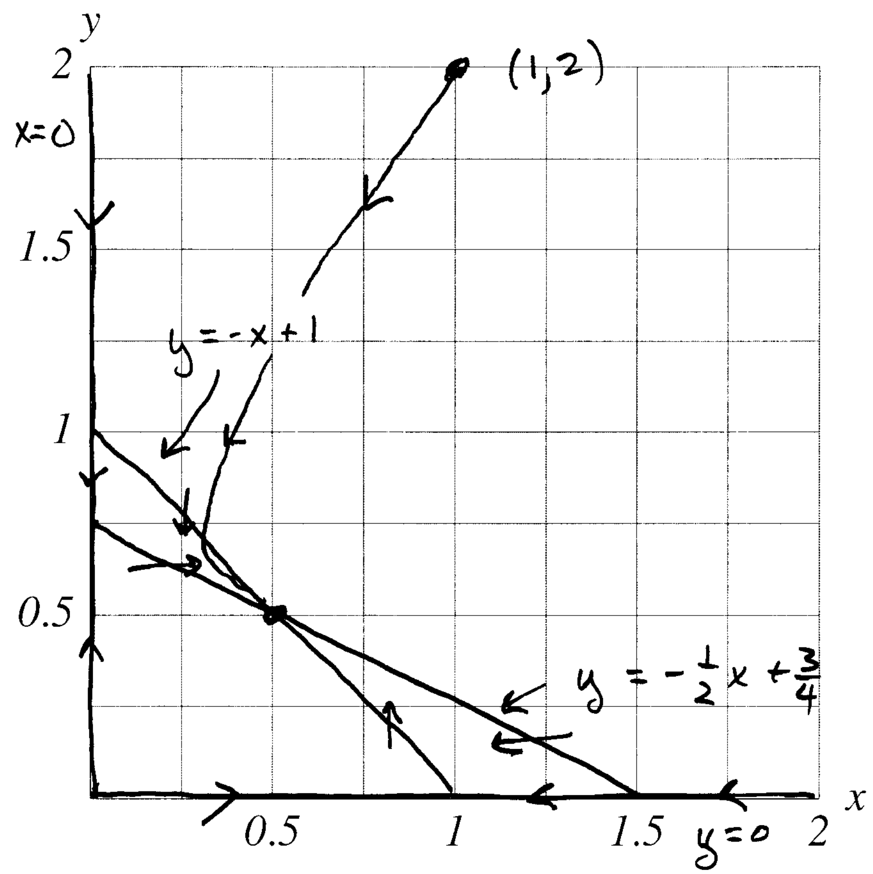
$$D\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}$$

Trace $= -1$ Determinant $= \frac{1}{8}$

$\Rightarrow$ $\left(\frac{1}{2}, \frac{1}{2}\right)$ is STABLE

$\Rightarrow$ Therefore, $(x(t), y(t)) \to \left(\frac{1}{2}, \frac{1}{2}\right)$ as $t \to \infty$ for trajectories near $\left(\frac{1}{2}, \frac{1}{2}\right)$. In other words, the two species can survive together.

(d) Sketch and *label* the nullclines on the graph below. Be sure to indicate the direction of the solution on the nullclines. Sketch the trajectory in xy-plane that begins at $(1, 2)$.

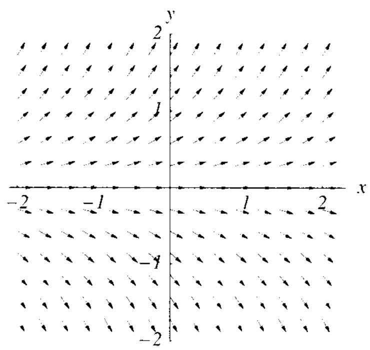


2. (6 points) Eight differential equations and six slope fields are given below. Determine the equation that corresponds to each slope field. No explanation is necessary. Each slope field is graphed for $-2 \le x \le 2$, $-2 \le y \le 2$.

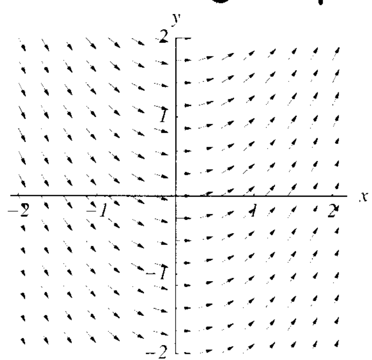
(a) $y' = \sin(2x)$ (c) $y' = 1 - y$ (e) $y' = x$ (g) $y' = y$

(b) $y' = 1 + x^2$ (d) $y' = 1 + y^2$ (f) $y' = y^2$ (h) $y' = x - y$

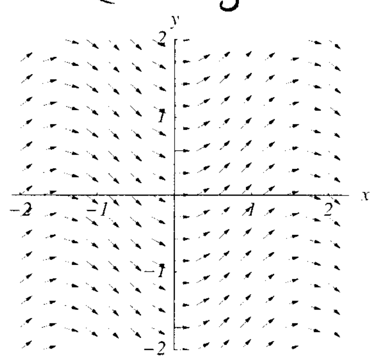
(i) (g) $y' = y$



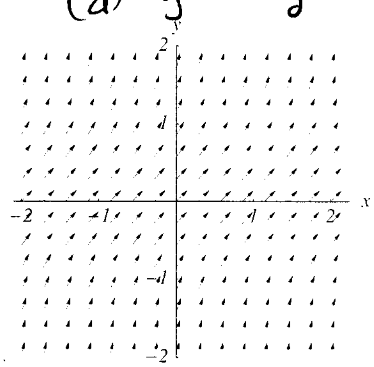
(iii) (e) $y' = x$



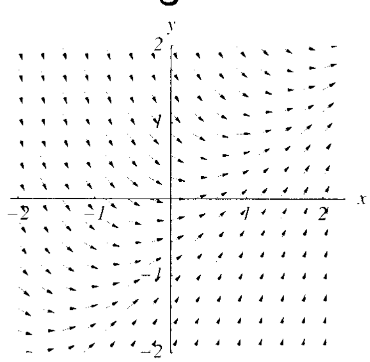
(v) (a) $y' = \sin(2x)$



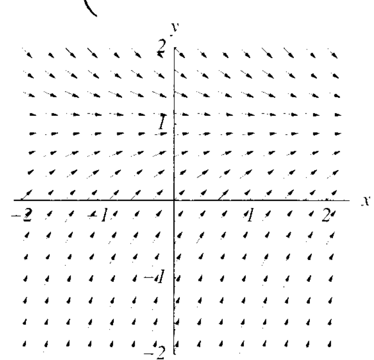
(ii) (d) $y' = 1 + y^2$



(iv) (h) $y' = x - y$



(vi) (c) $y' = 1 - y$



3. Compute each of the following partial derivatives: (8 points)

(a) $\frac{\partial}{\partial x}(x^2 + y^2 + x \sin y)$

$2x + \sin y$

(b) $\frac{\partial^2}{\partial x \partial y}(x^2 + y^2 + x \sin y)$

$\cos y$

(c) $\frac{\partial^2}{\partial x^2}(x^2 + y^2 + x \sin y)$

2

(d) $\frac{\partial^2}{\partial y^2}(x^2 + y^2 + x \sin y)$

$2 - x \sin y$

4. (10 points) Consider the following matrices.

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix}, B = \begin{pmatrix} -2 & 0 \\ 2 & 4 \end{pmatrix}, C = \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

Compute each of the following expressions.

(a) Find the trace of A

0

(b) Find the determinant of B.

8

(c) Calculate AB.

$$\begin{pmatrix} -6 & -8 \\ -8 & -4 \end{pmatrix}$$

(d) Calculate BA.

$$\begin{pmatrix} -2 & 4 \\ 14 & -8 \end{pmatrix}$$

(e) Calculate BC

$$\begin{pmatrix} -8 \\ -4 \end{pmatrix}$$

5. (10 points) Consider the following differential equation for the function $y(t)$:

$$\frac{dy}{dt} = y(y-5)(y-10)^2.$$

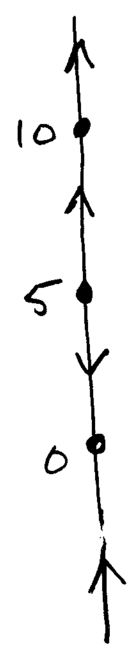
(a) What are the equilibrium solutions of this equation?

$y = 0$

$y = 5$

$y = 10$

(b) Graph the phase line for this equation.



(c) Classify each equilibrium solution as a source, a sink, or a node.

$y = 0$ is a sink

$y = 5$ is a source

$y = 10$ is a node

(d) If $y(0) = 4$, what happens as t gets very large?

$y(t) \to 0$

(e) If $y(0) = 6$, what happens as t gets very large?

$y(t) \to 10$

6. (10 points) The fox squirrel is a small mammal native to the Rocky Mountains. These squirrels are very territorial, so if their population is large, their growth rate decreases and will even become negative if the population is too large. On the other hand, if the population is too small, fertile adults run the risk of not being able to find suitable mates, so again the rate of growth is negative. Let

$$\begin{aligned} t &= \text{time (independent variable)}, \\ S(t) &= \text{population of the squirrels at time } t \text{ (dependent variable)}. \end{aligned}$$

Write a differential equation that models the squirrel population based on these assumptions.

$$\frac{dS}{dt} = kS\left(1 - \frac{S}{N}\right)\left(\frac{S}{M} - 1\right),$$

where

k = growth rate coefficient

N = carrying capacity

M = sparsity constant.

We must have a phase line of the form

7. (12 points) Consider the following predator-prey systems

(i)

$$\begin{aligned}\frac{dx}{dt} &= -\alpha x + \beta xy \\ \frac{dy}{dt} &= \gamma y - \delta xy\end{aligned}$$

(ii)

$$\begin{aligned}\frac{dx}{dt} &= \alpha x - \alpha\frac{x^2}{N} - \beta xy \\ \frac{dy}{dt} &= \gamma y + \delta xy.\end{aligned}$$

Assume that the parameters α, β, γ, δ, and N are all positive constants.

(a) Identify which dependent variable, x or y, is the predator population and which variable is the prey population in each case.

(i) x is the predator and y is the prey

(ii) x is the prey and y is the predator

(b) Is the growth of the prey limited by any factors other that the number of predators in each case?

(i) No limits on growth

(ii) Logistic growth

(c) In each case, do the predators have sources of food other than the prey?

(i) No alternative food source

(ii) Other alternative food sources.

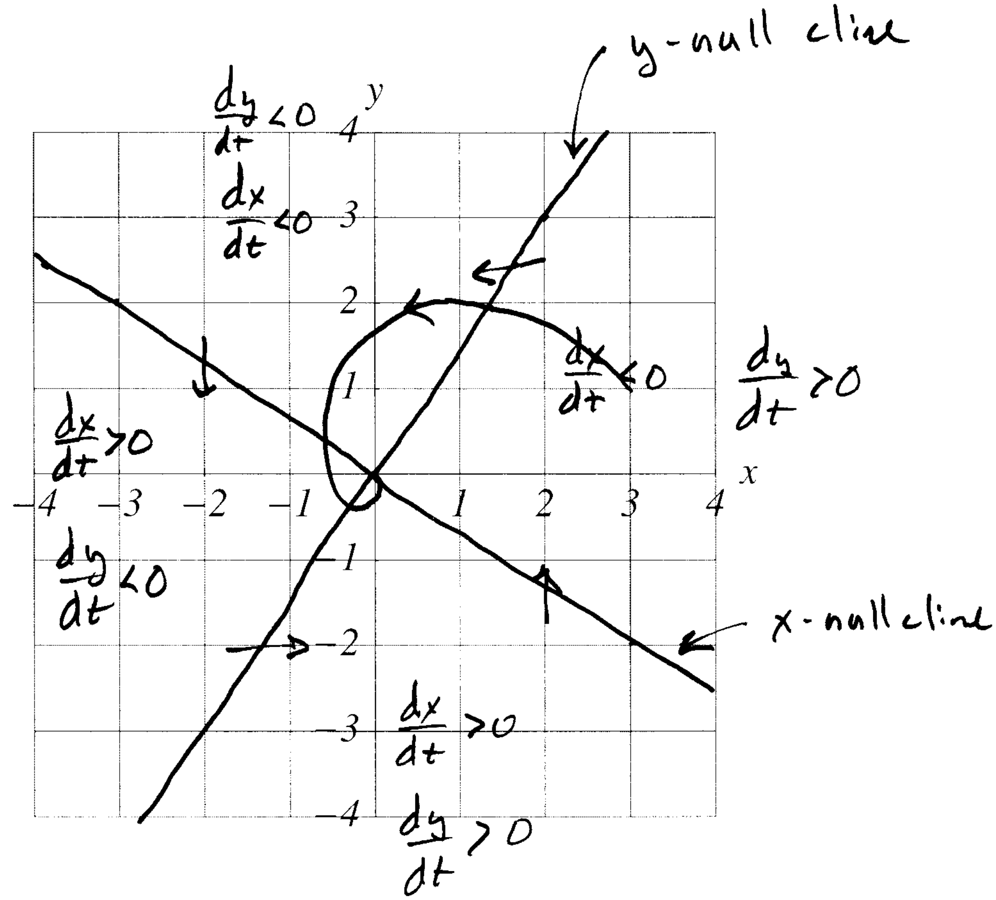
8. (11 points) Consider the system of differential equations

$$\frac{dx}{dt} = -2x - 3y,$$
$$\frac{dy}{dt} = 3x - 2y.$$

(a) Draw and label the x and y-nullclines on the graph below. Be sure to indicate the direction of the solution on the nullclines.

$$\frac{dx}{dt} = -2x - 3y = 0 \Rightarrow y = -\frac{2}{3}x \quad (x\text{-nullcline})$$

$$\frac{dy}{dt} = 3x - 2y = 0 \Rightarrow y = \frac{3}{2}x \quad (y\text{-null cline})$$



(b) Label the regions where $dx/dt > 0$ and where $dx/dt < 0$ on the graph on the previous page. Do the same for dy/dt.

SEE PREVIOUS PAGE.

(c) Decide if $(0, 0)$ is a stable equilibrium point. Justify your answer.

$$\begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}$$

Trace $= -4$

determinant $= 13$ $\Rightarrow$ STABLE

(d) For the initial condition $x(0) = 3$ and $y(0) = 1$, sketch the trajectory in the phase plane on the graph on the previous page.

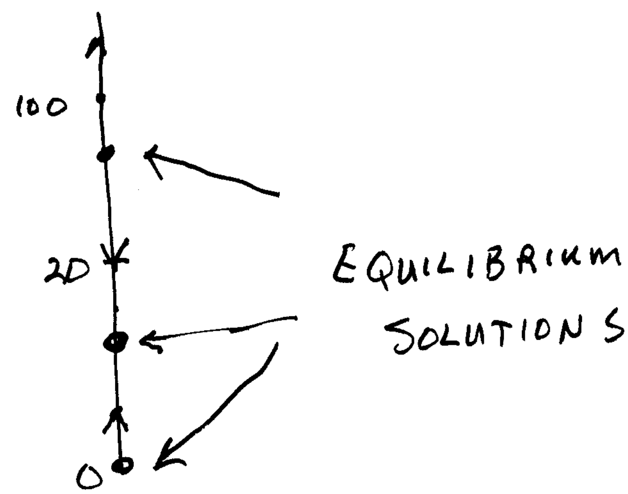
SEE PREVIOUS PAGE.

9. (8 points)

(a) Suppose that you wish to model a population with a differential equation of the form $dP/dt = f(P)$, where $P(t)$ is the population at time t. Experiments have been performed on the population that give the following information:

- The population at $P = 0$ remains constant.
- A population close to 0 will increase.
- A population of $P = 20$ will decrease.
- A population of $P > 100$ will increase.

Sketch the simplest possible phase line that agrees with the experimental information above.



(b) Consider the differential equation $dy/dt = f(y)$, where the graph of $f(y)$ is given below. Sketch the phase line for this differential equation.