Math 19. Lecture 21
Pattern Formation (I)

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1 The Model

- Suppose that the yellow color of tiger hair is caused by a high concentration of a particular protein in certain cells. the black color is caused by low concentration of the same protein in other cells.

- Suppose that the chemical concentration in a tiger embryo is described by a function
$$u = u(t, x)$$
at time t, where x is the only spatial coordinate.
$$0 \leq x \leq L$$
$$t \geq 0$$

- The chemical moves in a random way through the embryo.

- We can model this with the reaction-diffusion equation
$$\frac{\partial u}{\partial t} = \underbrace{\mu \frac{\partial^2 u}{\partial x^2}}_{\text{diffusion term}} + \underbrace{f(u),}_{\text{reaction term}}$$
where, for example,
$$\begin{aligned} f(u) &= r_0 - r_1 u, \text{ where } r_0 > 0, \\ f(u) &= r_1 u - r_2 u^2, \text{ where } r_1, r_2 > 0. \end{aligned}$$

- In practice we choose f such that $dv/dt = f(v)$ for a function $v(t)$ that describes the amount of the protein as a function of time in a single isolated cell. We can then decide f by experiments on isolated cells.
- *Important Assumption.* The protein in question is spread by random motion.

2 Boundary Conditions

How do derive the equation and how do we impose boundary conditions upon u at $x = 0$ and $x = L$? Recall how we derived the advection equation. We will do something similar here.

- Consider the strip $a \leq x \leq a + \Delta x$. The total amount of the protein in this is strip is given by

$$m(t,a) = \int_a^{a+\Delta x} u(t,x)\,dx,$$

  and $m(t,a) \approx u(t,a)\Delta x$ when Δx is small.

- If

$$\begin{aligned} q(t,a) &= \text{(rate at which molecules at } x = a \text{ pass from left to right)} \\ &\quad -\text{(rate at which molecules at } x = a \text{ pass from right to left)}, \end{aligned}$$

  then

$$\Delta x \frac{\partial u}{\partial t} \approx \frac{\partial m}{\partial t} = q(t,a) - q(t,a+\Delta x) + \underbrace{\int_a^{a+\Delta x} f(u(t,s))\,ds}_{\text{produced by the reaction}} .$$

  or

$$\frac{\partial u}{\partial t} \approx -\frac{q(t,a+\Delta x) - q(t,a)}{\Delta x} + \frac{1}{\Delta x}\int_a^{a+\Delta x} f(u(t,s))\,ds.$$

  As $\Delta x \to 0$, we obtain the equation

$$\frac{\partial u}{\partial t} = -\frac{\partial q}{\partial x} + f(u).$$

- Since we assume that the protein moves randomly,

$$q(t,a) = -\mu \frac{\partial u}{\partial x}.$$

  Thus,

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + f(u).$$

- At $x = 0$ and $x = L$, there is no chemical passing from right to left or from left to right. Thus,

$$q(t,0) = q(t,L) = 0$$

  or

$$\frac{\partial}{\partial x} u(t,0) = \frac{\partial}{\partial x} u(t,L) = 0.$$

3 Equilibrium Solutions

Our goal is to find solutions to

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + f(u)$$

that are stable with respect to time ($\partial u/\partial t = 0$) subject to the boundary conditions

$$\frac{\partial}{\partial x} u(t,0) = \frac{\partial}{\partial x} u(t,L) = 0.$$

- Our interest in the equilibrium solution implies that the stripe pattern, once set, does not change over time.

- If u is only a function of x; i.e., $u(t,x) = u_e(x)$, then $\partial u/\partial t = 0$, and $u_e(x)$ must satisfy

$$\mu \frac{d^2 u_e}{dx^2} + f(u_e) = 0$$
$$\frac{d}{dx} u_e(0) = \frac{d}{dx} u_e(L) = 0.$$

- *From now on, think of u_e instead of u.*

4 Stability

Suppose that there is a solution $u_e(x)$ for some f, say $f(u) = r_1 u - r_2 u^2$, where $r_1, r_2 > 0$. Is there a reasonable chance of seeing this solution in nature?

Suppose that $u_e(x)$ is a solution to

$$\mu \frac{d^2 u_e}{dx^2} + f(u_e) = 0$$

subject to the boundary conditions. Let $w(x)$ be a small perturbation of $u_e(x)$ at $t = 0$, and set

$$u(0, x) = u_e(x) + w(x)$$

and move forward in time to obtain a solution to

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + f(u) \tag{1}$$

$$\frac{\partial}{\partial x} u(t, 0) = \frac{\partial}{\partial x} u(t, L) = 0. \tag{2}$$

that is equal to $u_e(x) + w(x)$ at $t = 0$

Condition for Stability. If $w(x)$ is small enough, then the resulting solution $u(t, x)$ to (1) and (2) that has the property $u(0, x) = u_e(x) + w(x)$ has the property that at *every* x, the values of $u(t, x) \to u_e(x)$ as $t \to \infty$.

Condition for Instability. A solution is unstable if there is an arbitrarily small (but not identically zero) perturbation $w(x)$ such that $u(t, x)$ does not approach $u_e(x)$ for *at least one* x as $t \to \infty$.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 18.

- "Dynanamics of Stripe Formation," pp. 280–282.

- "A Reaction-Diffusion Wave on the Skin of the Marine Angelfish," pp. 282–286.

- "Letters to Nature," pp. 286–288.