Math 19. Lecture 26
Traveling Waves

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1 A Model for the Hantavirus

Suppose that mice are infected with the hantavirus in California. We wish to model how fast the infected mice will disperse across the country. Here are our assumptions.

- Mice are infected with a virus that are harmless to them but virulent in people.
- Once a mouse is infected, it stays infected.
- Each mouse wanders over some small territory at random and meets uninfected mice. This is how the infection is spread.
- Infected mice pass the virus to some percentage of the uninfected mice that they encounter.
- At time $t = 0$, all of the mice in western California are infected, but mice in eastern California and the rest of the country are not infected.
- We will simplify our model of the United States by viewing the U.S. as an infinitely long strip whose topography and width are irrelevant. The x coordinate is very negative in San Francisco and very positive in Boston.

2 The Reaction-Diffusion Equation.

Let $u(t, x)$ denote the proportion of mice that are infected at time t and position x. We will make the assumption that $u(t, x)$ is controlled by an equation of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ru(1-u), \tag{1}$$

where $r > 0$ is a number whose value can be determined by studying the rate at which laboratory mice are infected by the virus. Equation (1) is called *Fisher's equation*.

3 The Traveling Wave Assumption

We might look for a solution to (1) that has the form of a traveling wave,

$$u(t, x) = f(x - ct),$$

where $c > 0$. Such a function maintains the same shape over time but is a wave that travels eastward at a constant speed c as $t \to \infty$. Thus,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t} f(x-ct) = -c\frac{df}{ds} \\
\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x} f(x-ct) = \frac{df}{ds} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2}{\partial x^2} f(x-ct) = \frac{d^2 f}{ds^2},
\end{aligned}$$

where $s = x - ct$. Substituting into (1), we get

$$-c\frac{df}{ds} = \frac{d^2 f}{ds^2} + rf(1-f)$$

Our first goal is to find a solution f to this equation that satisfies the following conditions:

1. $0 \leq f \leq 1$,
2. $f(s) \to 1$ as $s \to -\infty$,
3. $f(s) \to 0$ as $s \to \infty$.

Ideally, we wish to find a solution at $t = 0$. In this case, $u(0, x) = f(x)$.

4 A Standard Trick

We can turn any second-order differential equation such as

$$-c\frac{df}{ds} = \frac{d^2f}{ds^2} + rf(1-f). \tag{2}$$

into a first-order system by introducing a new variable p such that $p = f'$. That is,

$$\frac{df}{ds} = p \tag{3}$$

$$\frac{dp}{ds} = -cp - rf(1-f). \tag{4}$$

The f null cline is $p = 0$, while the p null cline is

$$p = -\frac{r}{c}f(1-f) = \frac{r}{c}f^2 - \frac{r}{c}f$$

Therefore, we have two equilibrium points: $f = 0, p = 0$ and $f = 1, p = 0$.

The stability matrices for the first and second equilibrium points are

$$A = \begin{pmatrix} 0 & 1 \\ -r & -c \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ r & -c \end{pmatrix}.$$

Since the trace of A is negative and the determinant is positive, the first equilibrium point, $(0,0)$, is stable. Since $\det(B) < 0$, the second equilibrium point, $(1,0)$, is unstable.

5 The Phase Plane Solution

We are interested in a solution to (4) that tends to the origin as $s \to \infty$ and goes to unstable equilibrium point, $(1,0)$, as $s \to -\infty$. We will show when $c^2 > 4r$, there is such a trajectory in the lower half of the phase plane.

- There is a trajectory in the $p < 0$ part of the phase plane that tends towards $(1,0)$ as $s \to -\infty$ and that for moderately negative values of s, has $p < 0$ and $f < 1$. Since f must be a decreasing function, we require $f' = p < 0$.

- When $c^2 \geq 4r$, then the triangle pictured below is a region (called a *basin of attraction* or a *trapping region*) for (3) and (4). We claim that any trajectory in this region must tend towards $(0,0)$ as $s \to \infty$. Once a trajectory enters this region, it can *never* escape. You can think of a basin of attraction as a black hole.

6 Some Observations

1. On the line segment where $p = 0$ and $0 < f < 1$, we know that $dp/ds < 0$, so every trajectory that crosses this line moves from above to below.

2. On the half-line where $p < 0$ and $f = 1$, we know that $df/ds < 0$, so every trajectory that crosses this half-line moves from right to left.

7 Trajectories that Leave $(1, 0)$

8 The Triangle as a Basis of Attraction

Consider the triangle where the hypotenuse is formed by the line $p+cf/2 = 0$. We wish to know how we can choose c so that our solution f will satisfy all of the conditions that we have specified.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 21.

- "Hantavirus Outbreak Yields to PCR," pp. 358–363

- "US Braces for Hantavirus Outbreak," pp. 363–364.