Math 19. Mathematical Modeling
Solutions to Exam II—Fall 2005
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1. (10 points) Consider the *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $0 \leq x \leq \pi$ and $t \leq 0$. ***Caution—This is not the diffusion equation.***

(a) Show that $u_n(t,x) = \cos(cnt)\sin(nx)$ is a solution to equation (1) for $n = 1, 2, 3, \ldots$.

Solution.

$$\frac{\partial^2 u_n}{\partial t^2} = -c^2 n^2 \cos(cnt)\sin(nx) = c^2 \frac{\partial^2 u_n}{\partial x^2}.$$

(b) Use the Principle of Superposition and the functions $u_n(t,x) = \cos(cnt)\sin(nx)$ to construct a solution to equation (1) that satisfies the boundary and initial conditions

$$\begin{aligned} u(0,x) &= 5\sin 2x - 2\sin 3x + \sin 4x, \\ u(t,0) &= u(t,\pi) = 0. \end{aligned}$$

Solution. If $u(t,x) = \alpha u_2 + \beta u_3 + \gamma u_4$, then

$$\begin{aligned} u(0,x) &= \alpha u_2(0,x) + \beta u_3(0,x) + \gamma u_4(0,x) \\ &= \alpha \sin(2x) + \beta \sin(3x) + \gamma \sin(4x) \\ &= 5\sin 2x - 2\sin 3x + \sin 4x. \end{aligned}$$

Thus,

$$u(t,x) = 5\cos(2ct)\sin 2x - 2\cos(3ct)\sin 3x + \cos(4ct)\sin 4x$$

2. (15 points) Consider the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u.$$

(a) Use the separation of variables technique to find nonzero solutions to this equation subject to the conditions for all $t \geq 0$ and for $0 \leq x \leq L$ such that $u(t,0) = u(t,L) = 0$.

Solution. If $u(t,x) = A(t)B(x)$, then

$$\frac{A'(t)}{A(t)} = \frac{B''(x)}{B(x)} + 2 = \lambda$$

and

$$A(t) = A(0)e^{\lambda t}.$$

The only nonzero solutions for $B(x)$ are given by

$$B(x) = \alpha \cos\left(\sqrt{-c}\, x\right) + \beta \sin\left(\sqrt{-c}\, x\right),$$

where $c = \lambda - 2$. Since $B(0) = 0$, we find that $\alpha = 0$. The condition

$$0 = B(L) = \beta \sin\left(\sqrt{-c}\, L\right)$$

tells us that we have nonzero solutions if $\sqrt{-c} = n\pi/L$ for $n = 1, 2, \ldots$. Since

$$c = \lambda - 2 = -\frac{n^2\pi^2}{L^2},$$

we must have

$$\lambda = 2 - \frac{n^2\pi^2}{L^2}.$$

Therefore, the solutions that we seek are

$$u(t, x) = e^{(2 - n^2\pi^2/L^2)t} \sin\left(\frac{n\pi}{L} x\right).$$

(b) Find a condition on L that guarantees a bounded solution with respect to time. That is, show that there is a solution $u = u(t, x)$ to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u$$

for all $t \geq 0$ and for $0 \leq x \leq L$ with $u(t, 0) = u(t, L) = 0$ such that

$$\lim_{t \to \infty} u(t, x) < \infty.$$

Solution. For the solution to be bounded

$$\lambda = 2 - \frac{n^2\pi^2}{L^2} \leq 0$$

Therefore,

$$L \leq \frac{n\pi}{\sqrt{2}}.$$

We must choose n large enough for this inequality to hold.

3. (20 points) Consider the differential equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} - 2\sin(\pi u).$$

(a) Make the traveling wave substitution $u(t,x) = f(x-ct)$, where $c > 0$ is a constant to derive a differential equation in one variable, say s, for the function f.

Solution.

$$\frac{d^2 f}{ds^2} = -\frac{c}{2}\frac{df}{ds} + \sin(\pi f)$$

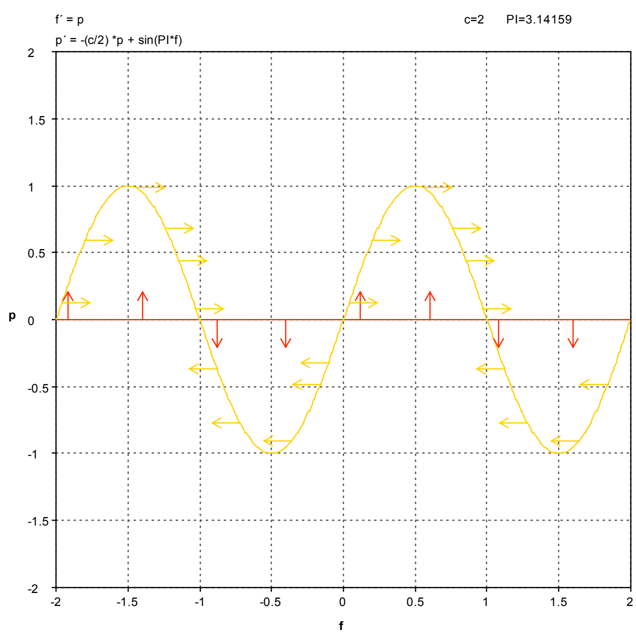
(b) Rewrite the equation that you derived in part (a) as a pair of autonomous differential equations in the variables f and p, where both of these functions depend on the independent variable s.

Solution.

$$\begin{aligned} \frac{df}{ds} &= p \\ \frac{dp}{ds} &= -\frac{c}{2}p + \sin(\pi f) \end{aligned}$$

(c) Sketch the phase plane for the system that you derived in part (b). Be sure to label the f and p nullclines, the equilibrium points, and mark the nullclines with arrows to indicate the direction of the trajectories the cross them.

Solution.



(d) Classify the equilibrium solutions at $(0,0)$ and $(1,0)$. Is it possible for traveling wave solutions $u(t,x) = f(x-ct)$ (where $c > 0$) to exist that satisfy the following conditions?

- $0 \le f \le 1$,
- $f(s) \to 1$ as $s \to -\infty$,
- $f(s) \to 0$ as $s \to \infty$.

Justify your answer.

Solution. Let

$$D(f,p) = \begin{pmatrix} \frac{\partial}{\partial f} p & \frac{\partial}{\partial p} p \\ \frac{\partial}{\partial f}(-\frac{c}{2}p + \sin(\pi f)) & \frac{\partial}{\partial p}(-\frac{c}{2}p + \sin(\pi f)) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \pi cos(\pi f) & -c/2 \end{pmatrix}$$

Since

$$D(0,0) = \begin{pmatrix} 0 & 1 \\ \pi & -c/2 \end{pmatrix},$$

we know that $(0,0)$ is unstable.
Since

$$D(1,0) = \begin{pmatrix} 0 & 1 \\ -\pi & -c/2 \end{pmatrix},$$

we know that $(0,0)$ is stable. Since $(1,0)$ is stable, $f(s) \to 1$ as $s \to \infty$. Therefore, it is not possible for traveling wave solutions to exist satisfying the conditions

- $0 \le f \le 1$,
- $f(s) \to 1$ as $s \to -\infty$,
- $f(s) \to 0$ as $s \to \infty$.

4. (10 points) Consider the equilibrium solution $u_e = 0$ to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(u+3)$$

satisfying the boundary conditions

$$u(t,0) = u(t,L) = 0.$$

Find a condition on L for $u_e(x) = 0$ to be a linearly stable solution.

Solution. If $f(u) = u^2 + 3u$ and $u_e = 0$, we have $f'(u_e(x)) = 3$. We can rewrite the equation

$$\lambda g = \frac{d^2 g}{dx^2} + 3g.$$

as

$$\frac{d^2 g}{dx^2} = cg,$$

where $c = \lambda - 3$. This equation has solution

$$g(x) = \alpha \cos\left(\sqrt{-c}\, x\right) + \beta \sin\left(\sqrt{-c}\, x\right).$$

The boundary condition $g(0) = 0$ tells us that $\alpha = 0$. Thus,

$$g(x) = \beta \sin\left(\sqrt{-c}\, x\right).$$

Since $g(L) = \beta \sin(\sqrt{-c}\, L) = 0$, we have a nonzero solution if $\sqrt{-c}\, L = n\pi$ or

$$c = \lambda - 3 = -\frac{n^2\pi^2}{L^2}$$

Thus,

$$\lambda = 3 - \frac{n^2\pi^2}{L^2}.$$

This constant is positive if

$$L \geq \frac{n\pi}{\sqrt{3}} > \frac{\pi}{\sqrt{3}}.$$

Thus, $u_e(x) = 0$ is not a stable solution if $L > \pi/\sqrt{3}$.

5. (10 points) Given the equation

$$\lambda g = \frac{d^2 g}{dx^2} - (\sin x) g,$$

use the Maximum Principle to determine if there is a positive number λ and a solution g defined on the interval $0 \leq x \leq \pi$ that vanishes at $x = 0$ and $x = \pi$ and is not identically zero. In particular, write *no* and justify your answer using the Maximum Principle or explain why the Maximum Principle cannot be applied.

Solution. We can rewrite the differential equation as

$$\frac{d^2 g}{dx^2} = (\lambda + \sin x) g.$$

If g has a positive maximum on $[0, \pi]$, then $g > 0$ and $g'' < 0$. However, $\lambda + \sin x > 0$ on this interval for positive λ, which tells us that $g'' > 0$ on this interval. This contradicts the fact that g has a positive maximum.

Now assume that g has a negative minimum. Since $\lambda + \sin x > 0$ and $g < 0$, we have $g'' < 0$, which contradicts the fact that we have a minimum. Thus, there exists no positive number λ and solution g defined on the interval $0 \leq x \leq \pi$ that vanishes at $x = 0$ and $x = \pi$ and is not identically zero.

6. (10 points)

   (a) Consider an equation, $x(t)$, as a function of time, of the form

$$\frac{dx}{dt} = \frac{1}{5}x^5 - x^3 - 4x + c,$$

   where c is a constant. Find all values of c where the number of equilibrium solutions changes.

Solution. Let

$$f(x) = \frac{1}{5}x^5 - x^3 - 4x + c.$$

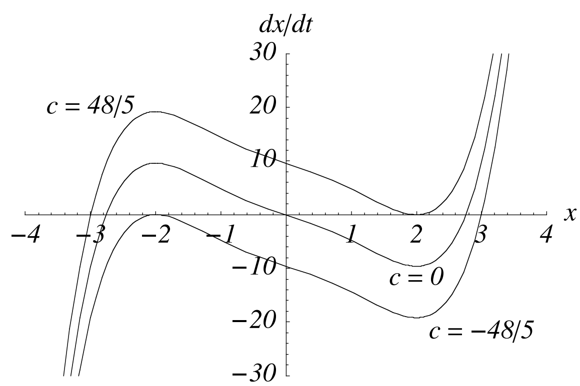
Then

$$f'(x) = x^4 - 3x^2 - 4 = (x-2)(x+2)(x^2+1).$$

Thus, f has critical points at $x = -2$ and $x = 2$. In fact, we have a maximum at $x = -2$ and a minimum at $x = 2$. The number of equilibrium solutions will change when the maximum or the minimum occur on the x-axis. The maximum occurs on the x-axis when

$$0 = f(-2) = 48/5 + c.$$

or when $c = -48/5$. Similarly, the minimum occurs on the x-axis when $c = 48/5$.



(b) As c increases, how does the stability of the equilibrium solutions change? In other words, explain how many stable and unstable equilibrium solutions there are for all values of c.

Solution. For $c < -48/5$, there is a single unstable solution. At $c = -48/5$, there are two unstable solutions. For $-48/5 < c < 48/5$, there are two unstable solutions and one stable solution. At $c = 48/5$, there are two unstable solutions. For $c > 48/5$, there is a single unstable solution.

7. (25 points) Consider the system

$$\begin{aligned} \frac{dx}{dt} &= 2x - y - x^3 \\ \frac{dy}{dt} &= x. \end{aligned}$$

(a) Show that $(0, 0)$ is a repelling equilibrium point for this system.

Solution. It is clear that $(0, 0)$ is an equilibrium solution for the system. To determine stability, observe that

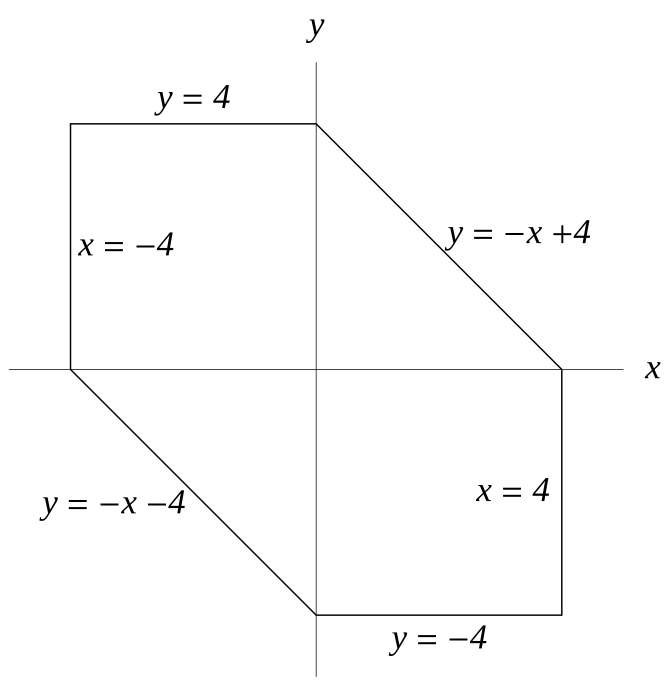
$$D(x, y) = \begin{pmatrix} \dfrac{\partial}{\partial x}(2x - y - x^3) & \dfrac{\partial}{\partial y}(2x - y - x^3) \\ \dfrac{\partial}{\partial x}(x) & \dfrac{\partial}{\partial y}(x) \end{pmatrix} = \begin{pmatrix} 2 - 3x^2 & -1 \\ 1 & 0 \end{pmatrix}.$$

At $(0, 0)$, we have

$$D(0, 0) = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since the trace and the determinant of this matrix are both positive, we know that $(0, 0)$ is an equilibrium solution for the system.

(b) Show that the hexagon S show below is a basin of attraction. Hence, the region S contains a closed orbit by the Poincaré-Bendixson Theorem. [***Hint***: By symmetry you need only check three sides of the hexagon instead of all six. Don't forget to check the corners.]



Solution. On the line $y = 4$, we know that $-4 \leq x \leq 0$. Thus,

$$\frac{dy}{dt} = x \leq 0,$$

and any trajectory must pass downward through the line $y = 4$.

At the corner $(-4, 4)$, any trajectory must pass into the basin of attraction since

$$\begin{aligned} \frac{dx}{dt} &= 52 \\ \frac{dy}{dt} &= -4. \end{aligned}$$

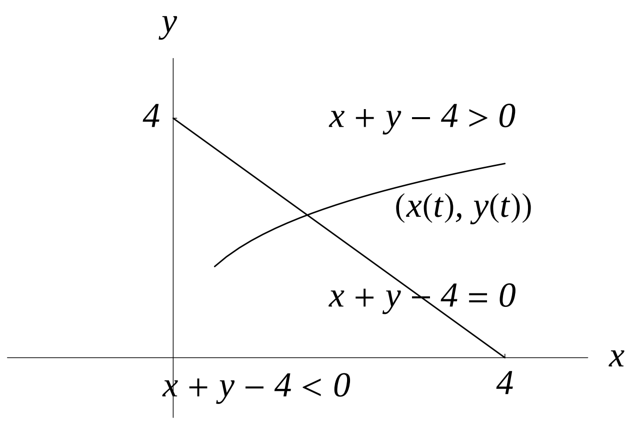
On the line $x = 4$,

$$\frac{dx}{dt} = y - 56.$$

Since $-4 \leq y \leq 0$, this derivative is always negative. Therefore, any trajectory must pass from right to left, and no trajectory can leave the basin of attraction. Now, we will consider trajectories $(x(t), y(t))$ through the line $x + y - 4 = 0$. Above the line, we know that $x + y - 4 > 0$, while below the line $x + y - 4 < 0$. Thus, we must show that

$$\frac{d}{dt}(x + y - 4) \leq 0$$

for any trajectory $(x(t), y(t))$ through the line $x + y - 4 = 0$.



However,

$$\begin{aligned} \frac{d}{dt}(x + y - 4) &= \frac{dx}{dt} + \frac{dy}{dt} \\ &= 2x - y - x^3 + x \\ &= 3x - y - x^3 \\ &= 3x - (-x + 4) - x^3 \\ &= 4x - 4 - x^3. \end{aligned}$$

We will show that $f(x) = 4x - 4 - x^3$ is negative for $0 \leq x \leq 4$. Since $f'(x) = 4 - 3x^2$, we know that f has a critical point at $x = 2/\sqrt{3}$. Thus, $f(2/\sqrt{3}) \approx -0.920799$ is a maximum since $f''(x) = -6x \leq 0$ on the interval $0 \leq x \leq$

4. Furthermore, this is an absolute maximum since f is concave down on this interval. Consequently, any trajectory through this line must pass into the basin of attraction.

Since $dx/dt = -56$ at $(4, 0)$ and any trajectory must pass downward through the line $x + y - 4 = 0$, no trajectory can leave the corner $(4, 0)$.

Since $dy/dt = 0$ at $(0, 4)$ and any trajectory must pass downward through the line $x + y - 4 = 0$, no trajectory can leave the corner no trajectory can leave the corner $(0, 4)$.

By symmetry, we can conclude that no trajectory can leave the basin of attraction through the remaining corners or edges.