Math 19. Name________________

Mathematical Modeling
Exam II—Fall 2003
T. Judson

Do not write in this space.

Problem Number	Possible Points	Score
1	5	
2	10	
3	15	
4	10	
5	10	
6	15	
7	5	
8	5	
9	10	
Total	85	

Directions—Please Read Carefully! You have two hours to take this midterm. Make sure to use correct mathematical notation. Any answer in decimal form must be accurate to three decimal places, unless otherwise specified. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You do not have to answer the problems in any particular order, so move to another problem if you find you are stuck or spending too much time. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit (except if the directions for that question specifically say no justification is necessary, such as in a True/False section). Please be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. You may use a calculator on this exam, and you are permitted one 4×6 inch note card. No other aids are allowed. ***Good Luck!!!***

1. (5 points) In each of the following cases, indicate which single model would be *most* appropriate: advection, diffusion, or a model using Laplace's equation. No justification is necessary.

   (a) The distribution of a certain protein in a cell after the protein has reached a steady-state.

   (b) The spread of airborne radioactivity after Chernobyl disaster in 1986.

   (c) The spread of the oil spill from the Exxon Valdez in Prince William Sound in 1989.

   (d) The spread of an anthrax epidemic in livestock.

   (e) The concentration of a drug in the bloodstream after it is injected into the arm.

2. (10 points) Consider the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $0 \le x \le \pi$ and $t \ge 0$. It is easy to verify that $u_1(t,x) = e^{-t}\sin x$ and $u_2(t,x) = e^{-4t}\sin 2x$ are solutions to (**??**).

(a) Show that $u_3(t,x) = e^{-9t}\sin 3x$ is also a solution to (**??**).

(b) Use the Principle of Superposition and the functions u_1, u_2, and u_3 to construct a solution to (**??**) that satisfies the boundary and initial conditions

$$u(0,x) = 60\sin x + 5\sin 2x - 20\sin 3x,$$
$$u(t,0) = u(t,\pi) = 0.$$

3. (15 points) Consider the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 7u.$$

(a) Use the separation of variables technique to find nonzero solutions to this equation subject to the conditions for all t and for $0 \leq x \leq L$ such that $u(t, 0) = u(t, L) = 0$.

(b) Find the minimum positive number L with the following property: There is a solution to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 7u$$

for all t and for $0 \leq x \leq L$ such that $u(t,0) = u(t,L) = 0$ and such that grows in size as $t \to \infty$.

4. (10 points) Consider the equilibrium solution $u_e = 3$ to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2 - 3u$$

satisfying the boundary conditions

$$\frac{\partial}{\partial x}u(t,0) = \frac{\partial}{\partial x}u(t,L) = 0.$$

Determine the stability of this solution.

5. (10 points) Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 5u,$$

where $0 \leq x \leq 1$ and $u(t, 0) = u(t, 1) = 0$. Is it possible to find a solution subject to these boundary equations that grows in size as $t \to \infty$?

6. (15 points) Consider the system

$$\begin{aligned}\frac{dx}{dt} &= x - 2y - (x^2 + y^2)x \\ \frac{dy}{dt} &= 2x + y - (x^2 + y^2)y.\end{aligned}$$

(a) Show that the origin is an equilibrium point of the system.

(b) Prove that zero is an repelling equilibrium point.

(c) Show that the square

$$R = \{(x, y) : -2 \leq x \leq 2, -2 \leq y \leq 2\}.$$

is a basin of attraction for the system

$$\begin{aligned} \frac{dx}{dt} &= x - 2y - (x^2 + y^2)x \\ \frac{dy}{dt} &= 2x + y - (x^2 + y^2)y. \end{aligned}$$

Hence, we can determine that the system has a period solution inside R by the Poincaré-Bendixson Theorem.

7. (5 points) Given the differential equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} - 6\frac{\partial u}{\partial x} + u^3,$$

make the traveling wave substitution $u(t,x) = f(x - ct)$, where $c > 0$ is a constant to derive a differential equation in one variable for the function f.

8. (5 points) Given the second order differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 7x(1-x) = 0,$$

write an equivalent first-order system of differential equations.

9. (10 points) Given the equation

$$\lambda g = \frac{d^2 g}{dx^2} - (\cos x) g,$$

use the Maximum Principle to determine if there is a positive number λ and a solution g defined on the interval $0 \leq x \leq 1$ that vanishes at $x = 0$ and $x = 1$ and is not identically zero. In particular, write *no* and justify your answer using the Maximum Principle or explain why the Maximum Principle cannot be applied.