Midterm 2 review sheet

The point of this review sheet is to provide you with a quick summary of some of the formulas and definitions. You should already know this, and having them written down like this will save the time of going through them on the blackboard, thus allowing more emphasis on problem solving, which is the most important part of the exam. There will be references to this sheet in the problems that will be presented, so make sure you keep it at hand and follow on it everything that happens on the blackboard.

The problems are most of the time using material from various sections and you are definitely able to deal with that. Thus, the problem review will group the sections in an order dictated by the links between the sections, rather than strictly the one in the book. This review sheet is just summarizing the book, so it follows the order in there.

3.1 Derivatives of Polynomials and Exponential Functions

$\frac{d}{dx}(c) = 0$	Derivative of a Constant	3.1.1
$\frac{d}{dx}(x^n) = nx^{n-1}$, and in particular: $\frac{d}{dx}(x) = 1$	The Power Rule	3.1.2
$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$	The Constant Multiple Rule	3.1.3
$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$	The Sum Rule	3.1.4
$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x)-\frac{d}{dx}g(x)$	The Difference Rule	3.1.5
$\lim_{h\to 0}\frac{e^h-1}{h} = 1$	Definition of Number *e*	3.1.6
$\frac{d}{dx}(e^x) = e^x$	Derivative of the Natural Exponential Function	3.1.7

3.2 The Product and Quotient Rules

If *f* and *g* are both differentiable, then:

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$ The Product Rule 3.2.1

If *f* and *g* are both differentiable, then:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$ The Quotient Rule 3.2.2

3.3 Rates of Change in the Natural and Social Sciences

Average rate of change of *y* with respect to *x*, over the interval $[x_1, x_2]$ is $\frac{y(x_2)-y(x_1)}{x_2-x_1}$.

In the limit of x_1? x_2 we get the instantaneous rate of change of *y* with respect to *x*, which is exactly the derivative of *y* with respect to *x*, at that point, x_1.

3.4 Derivatives of Trigonometric Functions

$\lim_{q \to 0} \frac{\sin q}{q} = 1$ 3.4.1

$\frac{d}{dx}(\sin x) = \cos x$ 3.4.2

$\frac{d}{dx}(\csc x) = -\csc x \cot x$ 3.4.3

$\frac{d}{dx}(\cos x) = -\sin x$ 3.4.4

$\frac{d}{dx}(\sec x) = \sec x \tan x$ 3.4.5

$\frac{d}{dx}(\tan x) = \sec^2 x$ 3.4.6

$\frac{d}{dx}(\cot x) = -\csc^2 x$ 3.4.7

3.5 The Chain Rule

If f and g are both differentiable and $F = f \circ g$ is the composite function defined by *F(x)=f(g(x)),* then F is differentiable and:

F'(x)=f'(g(x))g'(x) The Chain Rule 3.5.1

If *y=f(u)* and *u=g(x)* are both differentiable, then: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ 3.5.2

Example: If n is any real number and *g(x)* is differentiable, then:

$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \bullet g'(x)$ The Chain Rule Combined with the Power Rule 3.5.3

$\frac{d}{dx}(a^x) = a^x \ln a$ 3.5.4

Tangents to Parametric Curves:

For a parametric curve defined by *x=f(t)* and *y=g(t),* use $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, if $\frac{dx}{dt} \neq 0$, in order to find the equation of the tangent.

3.6 Implicit Differentiation

Usually, the case is that of an implicit relationship between x and y. In order to find the derivative $\frac{dy}{dx}$, we take the derivative with respect to x of both sides of the equality, keeping in mind that y is a function of x, and applying the chain rule as needed. The resulting equality is to be then considered an equation in the unknown $\frac{dy}{dx}$, with coefficients depending on x and y.

Example from the book: $x^3 + y^3 = 6xy$. Differentiating each side with respect to x:

$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$, dividing by 3, moving all terms containing $\frac{dy}{dx}$ to the left side, and all the others to the right side and factoring $\frac{dy}{dx}$ on the left side, we get:

$\frac{dy}{dx}(y^2 - 2x) = 2y - x^2 \Rightarrow \frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$, the result we wanted.

Orthogonal Curves:
Two curves are orthogonal at the point given by (x_1, y_1) if the point is on both curves (x_1 and y_1 satisfy both equations of the curves) and the tangents at that point to the two curves are perpendicular. That is: $\frac{dy}{dx}$ (which depends on x and/or y) for $x=x_1$ and $y=y_1$, for the first curve, (call it m_1) is the negative reciprocal of the $\frac{dy}{dx}$ for $x=x_1$ and $y=y_1$, for the second curve, (call it m_2). Note: you plugged x_1 and y_1 in the general formula for each $\frac{dy}{dx}$ and you got the numbers m_1 and m_2. They have to satisfy $m_1 = -\frac{1}{m_2}$.

Derivatives of Inverse Trigonometric Functions:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad 3.6.1$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad 3.6.2$$

3.7 Derivatives of Logarithmic Functions

$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$, particularly: $\frac{d}{dx}(\ln x) = \frac{1}{x}$ 3.7.1

Combined with Chain Rule: $\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$ 3.7.2

Logarithmic differentiation:
1. Take natural logarithms of both sides of an equation $y=f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y' in terms of x and y.
4. Plug $y=f(x)$ back into the result from 3. in order to get y' only in terms of x.

$$\frac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)\, g'(x) \qquad 3.7.3$$

The number e as a limit:

$e = \lim_{x->0}(1+x)^{1/x}$ or $e = \lim_{n->\infty}\left(1+\frac{1}{n}\right)^n$, e˜ 2.71828182845904523536 (you don't need to know this ☺) 3.7.4

3.8 Linear Approximations and Differentials
If we don't know the general expression for a function *f*, but we know the function's value *f(a)* and the value of its derivative *f'(a)* for some point *a*, we can approximate *f* by its tangent at point *a* and thus find approximate values for *f(x)*, for *x* close to *a*.
Linear approximation: $f(x) \approx f(a) + f'(a)(x-a)$
Linearization: $L(x) = f(a) + f'(a)(x-a)$

Differentials:

The differential *dx* is an independent variable. Then, the differential $dy=f'(x)dx$.

4.1 Related Rates

1. Read the problem carefully.
2. Make a diagram if possible.
3. Introduce notation. Assign symbols to quantities depending on time.
4. Express given and required information in terms of derivatives.
5. Write an equation that relates the various quantities. If necessary try to eliminate one or more of the variables using geometry.
6. Differentiate both sides using Chain Rule.
7. Substitute what is known in the resulting equation and solve for the unknown rate.

4.2 Maximum and minimum values

A function has an absolute maximum (or global maximum) at c if $f(c)=f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the maximum value of f on D.
A function has an absolute minimum (or global minimum) at c if $f(c)=f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the minimum value of f on D.

A function has a local maximum at c if $f(c)=f(x)$ when x is near c.
A function has a local minimum at c if $f(c)=f(x)$ when x is near c.
Note 1: x near c means that x is in an open interval included in the domain of the function, D and containing c. The endpoints of the interval D cannot be contained in such an open interval. Thus, the endpoints of the interval D are never local minima or maxima, although they may be global ones.
Note 2: If a function is constant over a closed or open interval (horizontal line) than every point in that interval without the endpoints will be both a local maximum and minimum for the given initial closed or open interval.

The Extreme Value Theorem:

If f is continuous over a closed interval $[a,b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a,b]$.
Note: if the interval is not closed at both ends the theorem cannot be applied.

Fermat's Theorem:

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c)=0$.
Note: The reciprocal is not necessary valid. If $f'(c)=0$, it doesn't necessarily mean that c is a maximum or a minimum.

A critical number of a function f is a number c in the domain of f such that either $f'(c)=0$ or $f'(c)$ does not exist.

If f has a local maximum or minimum at c, than c is a critical number of f.
Finding absolute minimum/maximum of f on a closed interval $[a,b]$:
1. Find the values of f for all the critical numbers over the interval.
2. Find the values of f for the endpoints of the interval.
3. The smallest of the values from 1. and 2. is the global minimum, the largest is the global maximum.

4.3 Derivatives and the Shape of Curves

The Mean Value Theorem:

If f is differentiable on the interval [a,b], then there exists a number c between a and b such that $f'(c) = \frac{f(b) - f(a)}{b - a}$ 4.3.1

Increasing/Decreasing Test:

1. If $f'(x)>0$ on an interval, then f is increasing on that interval. 4.3.2
2. If $f'(x)<0$ on an interval, then f is decreasing on that interval. 4.3.3

The First Derivative Test:

If c is a critical number of a continuous function f:

1. If f' changes from positive to negative at c, then f has a local maximum at c. 4.3.4
2. If f' changes from negative to positive at c, then f has a local minimum at c. 4.3.5
3. If f' doesn't change sign at c, than f has no local maximum or minimum at c. 4.3.6

Concavity Test:

1. If $f''(x)>0$ for all x in I, then the graph of f is concave upward on I. 4.3.7
2. If $f''(x)<0$ for a all x in I, then the graph of f is concave downward on I. 4.3.8

The Second Derivative Test:

1. If $f'(c)=0$ and $f''(c)>0$, then f has a local minimum at c. 4.3.9
2. If $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at c. 4.3.10

4.5 Indeterminate Forms and L'Hospital's Rule

If f and g are differentiable and $g'(x)?0$ near a (except possibly at a) and that:

$\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$

or that $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

(we have an indeterminate of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$). Then:

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ L'Hospital's Rule 4.5.1

, if the limit on the right side exists.

Indeterminate Products:

Indeterminate form of type 0•8 (product fg, where f->0 and g->8).

Write $fg = \frac{f}{1/g}$ or $fg = \frac{g}{1/f}$ and apply L'Hospital.

Intermediate Differences:

Indeterminate form of type 8 -8 (f-g, where f->8 and g->8).

Try to get a common factor and/or find the common denominator and rationalize. Then use Indeterminate Products.

Indeterminate Powers:

For $\lim_{x\to a}[f(x)]^{g(x)}$:

1. $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$ type 0^0
2. $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=0$ type 8^{0}
3. $\lim_{x\to a} f(x)=1$ and $\lim_{x\to a} g(x)=\pm\infty$ type 1^8

Take the natural logarithm of both sides:

Let $y=[f(x)]^{g(x)}$ then $\ln y = g(x)\ln f(x) = \dfrac{\ln f(x)}{1/g(x)} = \dfrac{g(x)}{1/\ln f(x)}$ and

$\lim_{x\to a}(\ln y) = \ln(\lim_{x\to a} y)$ can be found using L'Hospital's Rule. 4.5.2

Writing the function as an exponential:

$[f(x)]^{g(x)} = e^{g(x)\ln f(x)}$ and then: $\lim_{x\to a} y(x) = \lim_{x\to a}[e^{g(x)\ln f(x)}] = e^{\lim_{x\to a}[g(x)\ln f(x)]}$ and the limit in the exponent can be found using L'Hospital's Rule, as above. 4.5.3

4.6 Optimization Problems

1. Understand the problem: What are you asked for? What are the given quantities? What are the given conditions?
2. Make a diagram. Identify those from above on the diagram. If you feel you need to, make more diagrams, for different values of the given changing quantity(ies), such that you get a picture of what is happening to the required quantity.
3. Introduce notation. Assign a symbol to the quantity to be maximized or minimized (call it *Q* for now). Assign symbols to the other unknown quantities. Label the diagram according to your convention. Use suggestive letters (*A* for area, *h* for height, *t* for time, etc.)
4. Express *Q* in terms of the symbols from 3.
5. If the result is a function *f* more than one variable, use the conditions given in the problem to express variables in terms of each other, such that you obtain *Q* in terms of only one of the symbols from 3., preferably the one you are mainly interested about in the problem (call it *x*). You now have a function *Q=f(x)*. Write its domain.
6. Use the methods you know for finding absolute maxima and minima to find the required extreme value for *f*. Use the Closed Interval Method only if the domain of *f* is a closed interval. If the domain is not a closed interval, the problem may not have a solution.

First Derivative Test for Absolute Extreme Values:

If *c* is a critical number of a continuous function *f* defined on an interval

1. If $f'(x)>0$ for all $x<c$ and $f'(x)<0$ for all $x>c$, then $f(c)$ is the absolute maximum value of *f*.
2. If $f'(x)<0$ for all $x<c$ and $f'(x)<0$ for all $x>c$, then $f(c)$ is the absolute minimum value of *f*.

GOOD LUCK!!!