Solutions to Math 1a Exam #1: Mon, Oct 25, 1999

(1) Find the following limits, if they exist:

(a) **Soln:** $\lim_{x\to 2}\frac{x^2+3x}{x^3+5x+1}=\frac{4+6}{8+10+1}=\frac{10}{19}$. In fact, most limits are calculated by simple evaluation.

(b) **Soln:** $\lim_{t\to -3}\frac{|t|}{|4t+6|}=\frac{3}{6}=\frac{1}{2}$

(c) **Soln:** $\lim_{w\to -1}\frac{3+4w+w^2}{1+w}=\lim_{w\to -1}\frac{(1+w)(3+w)}{1+w}=\lim_{w\to -1}(3+w)=2$

(2) Let $h(x)=\begin{cases}3x-2, & x\le 0\\ x^2-2, & x>0\end{cases}$

(a) Sketch the graph of h.
Soln: [See last page.]

(b) Find $\lim_{x\to 0}h(x)$ if it exists.

Soln: $\lim_{x\to 0^+}h(x)=-2$ and $\lim_{x\to 0^-}h(x)=-2$, so the limit exists and is equal to -2.

(c) Is h continuous? Why or why not?
Soln: YES. The only possible problem would be where the pieces patch together at $x = 0$, and at that point the limit exists and the value of the function is defined and equal to the limit. Hence it's continuous there and everywhere.

(d) Is h differentiable at $x = 0$? Why or why not?
Soln: NO. Approaching 0 from the left, the slope would be 3. Approaching from the right, the slope would be 0. Since the slopes do not match, the graph will not be smooth at $x = 0$ and hence will not have a well-defined tangent.

(3) a) Let $f(x) = x^2 + 2x$. What is the domain of f?
Soln: The domain is <u>all real numbers x</u>.

b) Find the equation of the tangent line at $x = 2$ using the definition of the derivative.

Soln:

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{[(2+h)^2+2(2+h)]-8}{h}$$

$$=\lim_{h\to 0}\frac{4+4h+h^2+4+2h-8}{h}=\lim_{h\to 0}\frac{h^2+6h}{h}=\lim_{h\to 0}(h+6)=6$$

So the slope is 6 and $f(2) = 8$. Hence the equation of the tangent line is $y - 8 = 6(x - 2)$.

4) Given the graph of the function $y = f(x)$ as shown, graph its first and second derivatives on the given axes, marking all noteworthy points appropriately.
Soln: [See last page.]

(5) Find derivatives of the following functions using any method.

(a) **Soln:** $f(x) = \frac{1}{\sqrt{x}}((\sqrt{x})^5 + 1) = x^2 + x^{-1/2}$

So $f'(x) = 2x - \frac{1}{2}x^{-3/2} = 2x - \frac{1}{2x\sqrt{x}}$.

(b) $f(x) = x^3 \cos 5x$

Soln: $f'(x) = x^3(-\sin 5x)(5) + 3x^2 \cos 5x = -5x^3 \sin 5x + 3x^2 \cos 5x$.

(c) $f(x) = \frac{\sin x}{3x+2}$

Soln: $f'(x) = \frac{(3x+2)\cos x - 3\sin x}{(3x+2)^2}$

(6) Find the equation of the tangent line to the curve $x^2y + 2y^3 = 3x + 2y + 54$ at the point where $(x, y) = (2,3)$.

Soln: Use implicit differentiation to find the derivative:

$$x^2 \frac{dy}{dx} + 2xy + 6y^2 \frac{dy}{dx} = 3 + 2\frac{dy}{dx}$$
$$(x^2 + 6y^2 - 2)\frac{dy}{dx} = 3 - 2xy$$
$$\frac{dy}{dx} = \frac{3-2xy}{x^2 + 6y^2 - 2}$$

Substituting in the point (2,3) gives that the slope is $-\frac{9}{56}$,

so the equation of the tangent line is $y - 3 = -\frac{9}{56}(x-2)$.

(7) A particle moves with displacement $s(t) = 16(1 - \frac{1}{t+1})$ meters.

(a) Find the average velocity from t = 1 to t = 3. **Soln**: Avg. velocity given by $\bar{v} = \frac{s(3)-s(1)}{3-1} = \frac{12-8}{2} = 2$

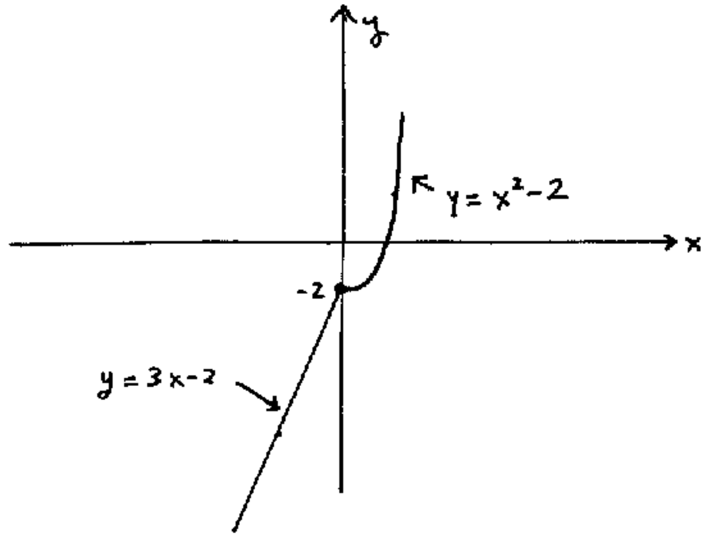
(b) Find the instantaneous velocity at t = 3. **Soln:** $v(t) = s'(t) = \frac{16}{(1+t)^2}$. So $v(3) = 1$.

8) Given that the side length of a cube is known to be 8 centimeters with a possible error of ± 2%, use linear approximation to estimate the percentage error in the surface area of the cube.

Soln: A cube has six faces and if we denote the side length by x, each face has area x^2. Thus the total surface area S is given by $S = 6x^2$. Using differentials, we have that $dS = 12x dx$. Since we want to look at percentage error (or relative error), it's best to divide through by the surface area: $\frac{dS}{S} = \frac{12x\,dx}{6x^2} = 2\frac{dx}{x}$.

Since the errors Δx and ΔS can be approximated by dx and dS, respectively, we have that the percent errors are related by $\frac{\Delta S}{S} = 2\frac{\Delta x}{x}$, so if the percent error in x is 2%, then the percent error in the surface area will be twice that, or 4%. The fact that the side length is 8cm is irrelevent.

PROBLEM 2 :



PROBLEM 4 :