SOLUTIONS TO MATH 1a FINAL EXAM – FALL 1999

(1a) $\boxed{\frac{3}{4}}$

(1b) $\lim\limits_{x\to k} \frac{x^3-kx^2}{x^2-k^2} \to \frac{0}{0}$

L'Hopital $\Rightarrow = \lim\limits_{x\to k} \frac{3x^2-2kx}{2x}$

$= \frac{3k^2-2k^2}{2k} = \frac{k^2}{2k} = \boxed{\frac{k}{2}}$

If $k=0$, then limit $= \underline{k}$

(1c) Look at

$\lim\limits_{x\to 0} \ln(1+\sin 2x)^{\frac{3}{x}}$

$= \lim\limits_{x\to 0} \frac{3\ln(1+\sin 2x)}{x} \to \frac{0}{0}$

$\therefore = \lim\limits_{x\to 0} \frac{3\cdot\frac{2\cos 2x}{1+\sin 2x}}{1} = 6$

$\therefore$ So limit $= \boxed{e^6}$

(2)(a) $x\cdot 2y\frac{dy}{dx} + y^2 + x^2\cdot\cos y\frac{dy}{dx} + 2x\sin y = 0$

$(2xy + x^2\cos y)\frac{dy}{dx} = -(y^2+2x\sin y)$

So $\boxed{\frac{dy}{dx} = -\frac{y^2+2x\sin y}{2xy+x^2\cos y}}$

(2b) $\frac{dy}{dx} = \frac{x(6e^{6x}+10x+\frac{1}{2}x^{-1/2}) - (e^{6x}+5x^2+\sqrt{x})}{x^2}$

$= \boxed{\frac{e^{6x}(6x-1)+5x^2-\frac{1}{2}\sqrt{x}}{x^2}}$.

(c) $\frac{d}{dx}\int_0^{\sin x}\sqrt{1-t^3}\,dt = \sqrt{1-\sin^3 x}\cdot\frac{d}{dx}(\sin x) = \cos x\sqrt{1-\sin^3 x}$

(d) $\ln y = \ln x^{x\sin x} = x\sin x\ln x$

$\frac{1}{y}\frac{dy}{dx} = 1\cdot\sin x\ln x + x(\sin x\,\frac{1}{x} + \cos x\ln x)$

$\Rightarrow \frac{dy}{dx} = x^{x\sin x}\left[\sin x\ln x + \sin x + x\cos x\ln x\right]$

(3)(a) $\int\frac{x\,dx}{\sqrt{x^2+1}} = \frac{1}{2}\int\frac{du}{u^{1/2}} = \frac{1}{2}\,\frac{u^{1/2}}{1/2}+C = \sqrt{x^2+1}+C$

$u = x^2+1$
$du = 2x\,dx$
$x\,dx = \frac{1}{2}du$

(3b) $\int(4-2\cos\theta)^3\sin\theta\,d\theta = -\int(4-2u)^3\,du$

$u = \cos\theta$
$-du = +\sin\theta\,d\theta$

$v = 4-2u$
$dv = -2du$
$-du = \frac{1}{2}dv$

$= \frac{1}{2}\int v^3\,dv$
$= \frac{1}{8}v^4 + C$
$= \frac{1}{8}(4-2u)^4 + C$
$= \frac{1}{8}(4-2\cos\theta)^4 + C$

(3c) $\int_5^{10}\frac{x\,dx}{\sqrt{x-1}} = \int_4^9\frac{(u+1)\,du}{u^{1/2}}$

$u = x-1$
$du = dx$
$x = u+1$

$= \int_4^9(u^{1/2}+u^{-1/2})\,du = \left.\frac{2u^{3/2}}{3}+\frac{2u^{1/2}}{1}\right]_4^9$

$= \left[\frac{2}{3}(27)+2\cdot 3\right] - \left[\frac{2}{3}(8)+2\cdot 2\right] = \frac{38}{3}+2 = \boxed{14\tfrac{2}{3}}$

(4) $f(x) = x^{1/4}$ $\quad f(16) = 2$ $\quad f(x) \cong f(16)+f'(16)(x-16)$

$f'(x) = \frac{1}{4}x^{-3/4}$ $\quad f'(16) = \frac{1}{32}$ $\quad \sqrt[4]{x} = 2+\frac{1}{32}(x-16)$

So $\sqrt[4]{15} \cong 2+\frac{1}{32}(-1) = \boxed{1\tfrac{31}{32}}$

(5) $S = 4\pi r^2 \qquad d = 2r \qquad d = 6 \pm .2$ cm (i.e. $d = 6$, $\Delta d = .2$ cm).

$dd = 2dr \Rightarrow \Delta d \cong 2\Delta r \Rightarrow \underline{\Delta r \cong .1\text{ cm}}$, $r = 3$ cm.

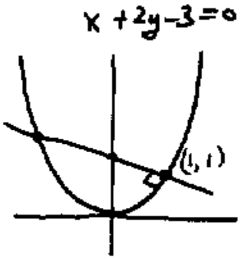
$\frac{dS}{dr} = 8\pi r \Rightarrow dS = 8\pi r\,dr$ or $\Delta S \cong 8\pi r \Delta r$.

This gives $\Delta S \cong 8\pi(3)(.1) = 2.4\pi$ (absolute) error.

Relative error is $\frac{\Delta S}{S} \cong \frac{8\pi r \Delta r}{4\pi r^2} = 2\frac{\Delta r}{r} = 2\left(\frac{.1}{3}\right) = \frac{2}{30} = \boxed{\frac{1}{15}}$.

(6) $\left.\begin{array}{l} y = x^2 \\ x + 2y - 3 = 0 \end{array}\right\} \Rightarrow$ Intersect where $y = (3-2y)^2 = 9 - 12y + 4y^2$

$\Rightarrow 4y^2 - 13y + 9 = 0 \Rightarrow y = \frac{13 \pm \sqrt{169 - 144}}{8} = \frac{13 \pm 5}{8}$



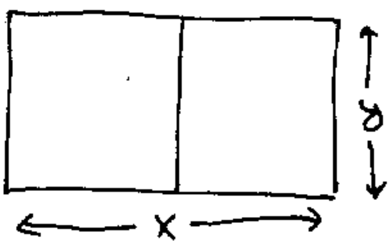

$\Rightarrow y = 1$ or $y = \frac{9}{4} \Rightarrow$ Points $(1,1)$ and $\left(-\frac{3}{2}, 1\right)$

$y = -\frac{1}{2}x + \frac{3}{2}$ SLOPE of TL for parabola given by $\frac{dy}{dx} = 2x$

SLOPE of LINE is $m = -\frac{1}{2}$

AT $(1,1)$, $\frac{dy}{dx} = 2$, so it's neg. recip. of slope of line $\Rightarrow$ $\boxed{\text{PERPENDICULAR AT } (1,1)}$.

(7)




$A = xy = 12000$ (area) $\Rightarrow y = \frac{12000}{x}$

Cost $C = 7.5(2x + 2y) + 3y = 15x + 18y$

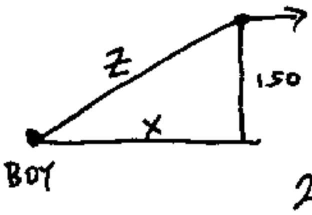
$C(x) = 15x + 18\left(\frac{12000}{x}\right)$ ← MINIMIZE THIS

$C'(x) = 15 - \frac{18 \cdot 12000}{x^2} = 0 \Rightarrow 15x^2 = 18 \cdot 12000 \Rightarrow x^2 = 18 \cdot 800$

$x^2 = 14400$

So $\boxed{x = 120} \Rightarrow y = \frac{12000}{120} = \boxed{100 = y}$ Easy to check that this gives a minimum.

(8)




$\frac{dx}{dt} = 20$ ft/sec $\qquad \frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} = \frac{200}{250}(20) = 16$ ft/sec

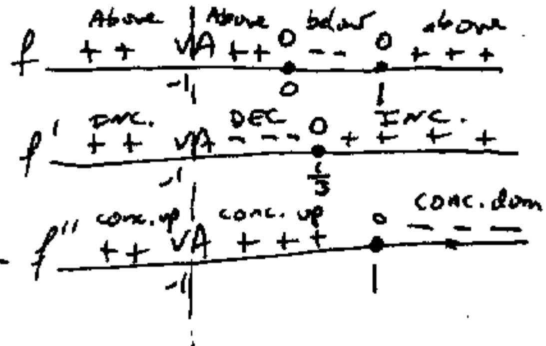
$x^2 + (150)^2 = z^2$

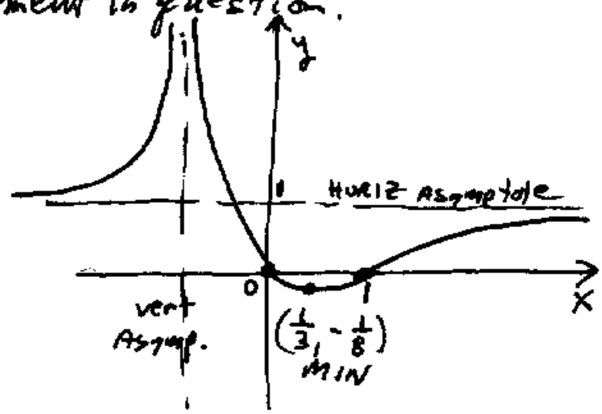
$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$ at moment in question.

(9) $f(x) = \frac{x(x-1)}{(x+1)^2}$

$f'(x) = \frac{3x-1}{(x+1)^3}$

$f''(x) = \frac{6(1-x)}{(x+1)^4}$



(10) $t=0$, $x=0$, $v=v_0$ $\longrightarrow$ $t=8$, $x=300$ ft, $v=50$ ft/sec

acceleration $a$ = constant

$$\frac{dv}{dt}=a \Rightarrow v(t)=at+v_0$$

$v(0)=v_0$ $\quad v(8)=\boxed{8a+v_0=50}$

$$\frac{dx}{dt}=at+v_0 \Rightarrow x(t)=\tfrac{1}{2}at^2+v_0t+x_0$$

$x(0)=x_0=0$ $\quad x(8)=\boxed{32a+8v_0=300}$

$$\begin{aligned}8a+v_0&=50\\32a+8v_0&=300\end{aligned} \Rightarrow \begin{aligned}64a+8v_0&=400\\32a+8v_0&=300\\\hline 32a\quad&=100\end{aligned} \Rightarrow \boxed{a=\frac{25}{8}\text{ ft/sec}^2}$$

$$v_0=50-8a=50-25 \quad \boxed{v_0=25\text{ ft/sec}}$$

(11) $f(x)=\dfrac{4x}{4-x}$

$$f'(x)=\frac{(4-x)4-4x(-1)}{(4-x)^2}=\frac{16}{(4-x)^2}$$

MVT: If $f$ is cont. on $[a,b]$ and diff'able on $(a,b)$ then there's a $c$ between $a$ and $b$ such that $\dfrac{f(b)-f(a)}{b-a}=f'(c)$

Here, $f$ is both continuous and differentiable on $[1,3]$. (The only potential problem is at $x=4$, but that's outside this interval.)

$$\frac{f(3)-f(1)}{3-1}=\frac{12-4/3}{2}=\frac{32}{6}=\frac{16}{3}=\frac{16}{(4-x)^2} \Rightarrow (4-x)^2=3$$

$$x-4=\pm\sqrt{3}$$

$$x=4\pm\sqrt{3}$$

$\sqrt{3}\cong 1.732$, so $4-\sqrt{3}\cong 2.268=c$ is within the interval $[1,3]$.

(12)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 3 | 10 | 29 | 66 |

$$\Delta x=\frac{4-1}{3}=1$$

(a) $L=[f(1)+f(2)+f(3)]\Delta x=(3+10+29)\,1=\boxed{42}$

(b) $R=[f(2)+f(3)+f(4)]\Delta x=(10+29+66)\,1=\boxed{105}$

(c) $T=\dfrac{L+R}{2}=\dfrac{42+105}{2}=\dfrac{147}{2}=\boxed{73.5}$

(d) $\displaystyle\int_1^4(x^3+2)\,dx=\left[\frac{x^4}{4}+2x\right]\Big|_{x=1}^{x=4}=(64+8)-(\tfrac{1}{4}+2)=\boxed{69.75}$

(e) $\bar{f}=\dfrac{\int_1^4(x^3+2)\,dx}{4-1}=\dfrac{69.75}{3}=\boxed{23.25}$