Fall 96

Math 1a – Second Midterm

[1] Sketch the graph of the function

$$f(x) = x^2 + \frac{2}{x}.$$

Label all intercepts, extrema, and points of inflection (give their co-ordinates), and pay attention to limits and asymptotes.

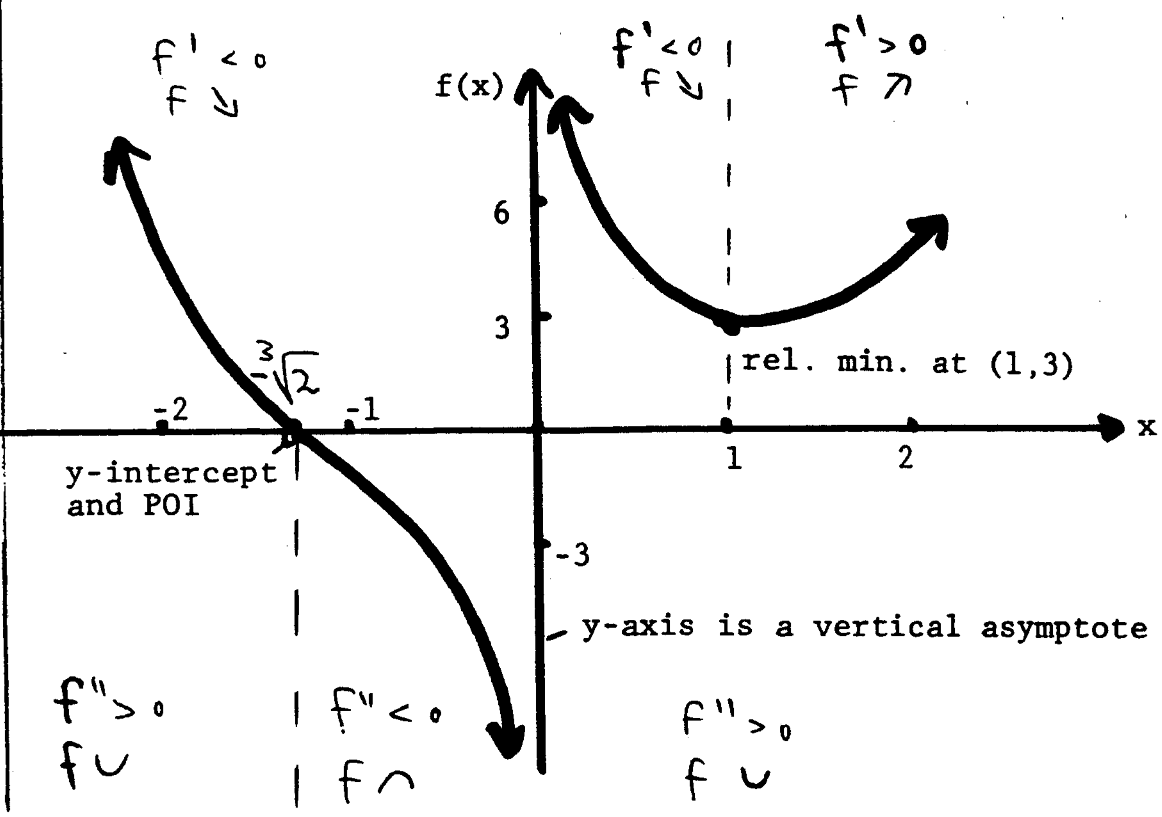
$f'(x) = 2x - \frac{2}{x^2}$

$f''(x) = 2 + \frac{4}{x^3}$

$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = \infty$

$\lim_{x \to 0^+} f(x) = \infty$

$\lim_{x \to 0^-} f(x) = -\infty$



[2] Find two positive numbers whose sum is 12 and such that the product of one of the numbers with the square of the other is as large as possible.

$P = x^2 y$

Constraint: $x + y = 12$, so $y = 12 - x$

Substitute: $P = x^2(12 - x) = 12x^2 - x^3$, $0 < x < 12$.

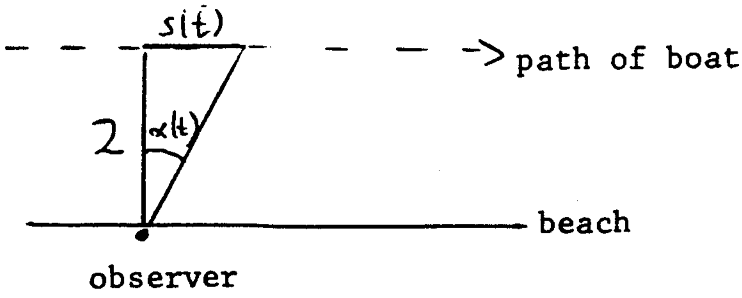
$P' = 24x - 3x^2 = 3x(8 - x) = 0$ For $\underline{x = 8}$

$P'' = 24 - 6x$ is negative for $x = 8$: we have a max.

$$\boxed{x = 8,\ y = 4}$$

3) An observer on a straight beach is watching a hovercraft which is travelling parallel to the shore, two miles out to sea, at a speed of 30 miles per hour. At what rate is the observer's head turning when the hovercraft passes directly in front of the observer?

"general situation"



$\tan(\alpha(t)) = \frac{1}{2} s(t)$ differentiate w.r to t

$\frac{1}{\cos^2(\alpha(t))} \alpha'(t) = \frac{1}{2} s'(t)$

$\alpha'(t) = \frac{1}{2}\cos^2(\alpha(t)) \cdot s'(t)$

At special instant $\alpha(t) = 0$, so $\cos^2(\alpha(t)) = 1$, $s'(t) = 30$

$\alpha'(t) = \frac{1}{2} \cdot 1 \cdot 30 = \boxed{15 \text{ (radians per hour)}}$

[4] (a) Find the following limit (if it exists):

(Use L'Hôpital's Rule twice)

$$\lim_{x \to 0} \frac{x^2}{1 - \cos(3x)}.$$

$$= \lim_{x \to 0} \frac{2x}{3\sin(3x)} = \lim_{x \to 0} \frac{2}{9\cos(x)} = \boxed{\frac{2}{9}}$$

(b) The graph of a certain function f has slope $\sin x + 7x^{12} + 4$ at every point (x, y) on the graph, and contains the point $(0, \pi)$. Find the function f.

$f'(x) = \sin x + 7x^{12} + 4$

$f(x) = -\cos x + \frac{7}{13} x^{13} + 4x + K$

Use the fact that $f(0) = \pi$ to find K:

$f(0) = -1 + K = \pi$, so that $k = \pi + 1$

$$\boxed{f(x) = -\cos(x) + \frac{7}{13} x^{13} + 4x + \pi + 1}$$

[4] (c) Let f be a function such that $f'(a)$ and $f''(a)$ exist at a point a. Find

$$\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h}.$$

chain rule

$$\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h} = \lim_{h\to 0}\frac{f'(a+h)+f'(a-h)}{2} = f'(a)$$

$f'(a)$ exists, so $f(x)$ is continuous at a, so $\lim_{h\to 0} f(a+h) = \lim_{h\to 0} f(a-h) = f(a)$. Hence, L'Hôpital's Rule applies.

$f''(a)$ exists, so $f'(x)$ is continuous at a, so $\lim_{h\to 0} f'(a+h) = \lim_{h\to 0} f'(a-h) = f'(a)$

It is more sensible to use the definition of the derivative to do this problem (think about it!)

[5] Consider a function $f(x)$ which is continuous on the closed interval $[p,q]$ and differentiable on the open interval (p,q). We are told that $f(p) = f(q) = 0$, and that the graph of $f(x)$ is concave down on (p,q). When answering the following questions you may use the Mean Value Theorem.

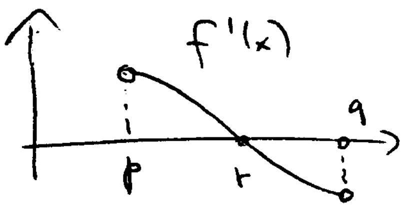
(a) Prove that the function $f(x)$ has a critical point r in the interval (p,q).

Apply <u>Rolle's Theorem</u> to the interval $[p,q]$

(b) What is the sign of $f'(x)$ if x is less than r? What if x is greater than r?

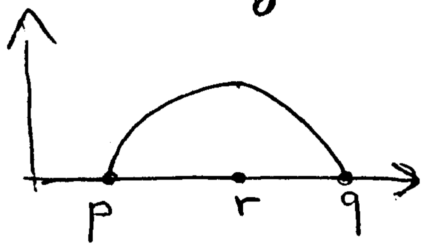
$f(x)$ is concave down, so that $f'(x)$ is decreasing. Therefore,

$$\boxed{\begin{array}{l} f'(x) > 0 \\ f'(x) < 0 \end{array}} \quad \begin{array}{l} \text{if } x < r \\ \text{if } x > r \end{array}$$



[5] (c) What can you say about the sign of $f(x)$ for x in (p, q)?

$f(x)$ is increasing on $[p, r]$ (since $f' > 0$), and
$f(x)$ is decreasing on $[r, q]$ (since $f' < 0$)



Therefore,
$f(x) > 0$ for x in $(p, r]$ (since $f(p) = 0$), and
$f(x) > 0$ for x in $[r, q)$ (since $f(q) = 0$).
We can conclude that
$\boxed{f(x) > 0}$ for x in (p, q)