

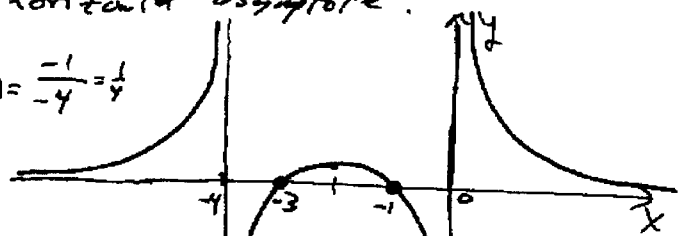
MATH 1a EXAM 2 REVIEW PROBLEM SOLUTIONS

- ①  $f(x)=0$  when  $x^2+4x+3=(x+3)(x+1)=0 \Rightarrow x=-3, x=-1$ . (x-intercepts)  
 $f'(x)=0$  when  $-6x-12=-6(x+2)=0 \Rightarrow x=-2$  (CRIT. PT.)  
 $f''(x)=0$  when  $3x^2+12x+16=0 \Rightarrow x=\frac{-12 \pm \sqrt{144-192}}{6} \Rightarrow$  never. (NO PTS OF INFLECTION)  
 Vertical asymptotes at  $x=0, x=-4$ .

$\lim_{x \rightarrow \pm\infty} f(x) = +1 \Rightarrow y=1$  a horizontal asymptote.

ABOVE	VA	BELOW	VA	ABOVE	BELOW	ABOVE
+	-	+	-	+	-	+
INC	-	DEC	-	INC	-	DEC
C. UP.	-	C. DOWN	-	C. UP.	-	C. UP.
+	-	+	-	+	-	+

$f(-2) = \frac{-1}{-4} = \frac{1}{4}$



② a)  $\lim_{x \rightarrow 1} \frac{x^4 - 5x^2 + 3x + 1}{x^3 - 2x + 1} \Rightarrow \frac{1-5+3+1}{1-2+1} = \frac{0}{0}$   
 $\therefore = \lim_{x \rightarrow 1} \frac{4x^3 - 10x + 3}{3x^2 - 2} = \frac{4-10+3}{3-2} = \boxed{-3}$

b)  $\lim_{x \rightarrow 0} \frac{\sin x - x}{(\cos x - 1)x} \rightarrow \frac{0}{0} \therefore = \lim_{x \rightarrow 0} \frac{\cos x - 1}{\cos x - 1 - x \sin x} \rightarrow \frac{0}{0}$   
 $\therefore = \lim_{x \rightarrow 0} \frac{-\sin x}{-2 \sin x - x \cos x} \rightarrow \frac{0}{0} \therefore = \lim_{x \rightarrow 0} \frac{-\cos x}{-3 \cos x + x \sin x} = \frac{-1}{-3} = \boxed{\frac{1}{3}}$

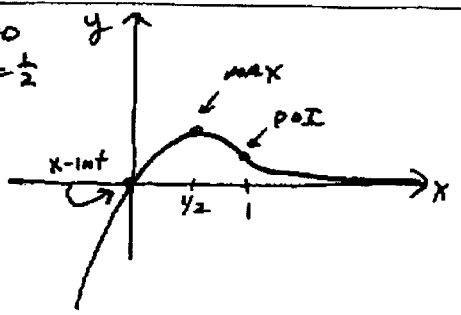
c) LOOK AT  $\lim_{x \rightarrow \infty} \ln \left( \frac{x+4}{x-1} \right)^x = \lim_{x \rightarrow \infty} x \ln \left( \frac{x+4}{x-1} \right) = \lim_{x \rightarrow \infty} \frac{\ln(x+4) - \ln(x-1)}{1/x} \rightarrow \frac{0}{0}$   
 $\therefore = \lim_{x \rightarrow \infty} \frac{\frac{1}{x+4} - \frac{1}{x-1}}{(-1/x^2)} = \lim_{x \rightarrow \infty} -x^2 \left[ \frac{(x-1) - (x+4)}{(x+4)(x-1)} \right] = \lim_{x \rightarrow \infty} \frac{5x^2}{x^2+3x-4} = \boxed{5}$   
 $\therefore \ln[\text{LIMIT}] = 5, \text{ SO LIMIT} = \boxed{e^5}$

③ a)  $f(x) = x^4 - 2, x_0 = 1, x_1 = N(x_0) = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{4} = \boxed{\frac{5}{4}}$   
 $f'(x) = 4x^3$

b)  $f(x) = x^3 - 7x - 5, x_0 = 3, x_1 = 3 - \frac{f(3)}{f'(3)} = 2.95, x_2 = 2.95 - \frac{f(2.95)}{f'(2.95)} \approx 2.9488 \dots$   
 $f'(x) = 3x^2 - 7$

④  $f(x) = x e^{-2x}$   
 $f'(x) = 2x e^{-2x} + e^{-2x} = e^{-2x}(1-2x)$   
 $f''(x) = -2e^{-2x} + (1-2x)e^{-2x}(-2) = e^{-2x}(4x-4)$   
 x intercept at  $x=0$   
 CRIT PT AT  $x=\frac{1}{2}$   
 P.O.I AT  $x=1$

x	0	1/2	1
f(x)	0	1/2e	1/e^2



⑤  $g(0) = 5, g(2) = 11$  MVT says there's a  $c$  between 0 and 2 such that  
 $g'(c) = \frac{g(2) - g(0)}{2 - 0} = \frac{11 - 5}{2} = \frac{6}{2} = 3$ , i.e. slope of TL = 3.

⑥ Area  $A = 3 \cdot y + 2(\frac{1}{2}xy) = 3y + xy$  and  $x^2 + y^2 = 9$   
 $A = y(3+x) = (3+x)\sqrt{9-x^2} = A(x) \quad 0 \leq x \leq 3$   
 $A'(x) = (3+x) \frac{1}{2}(9-x^2)^{-1/2}(-2x) + (9-x^2)^{1/2} = \frac{-x(3+x) + (9-x^2)}{\sqrt{9-x^2}} = \frac{9-3x-2x^2}{\sqrt{9-x^2}}$   
 CRIT POINT when  $2x^2 + 3x - 9 = 0 \Rightarrow x = \frac{-3 \pm \sqrt{9+72}}{4} = \frac{-3 \pm \sqrt{81}}{4} = \frac{-3 \pm 9}{4}$   
 $\Rightarrow x = \frac{3}{2}$  (in domain).  
 $A(0) = 9 \quad A(\frac{3}{2}) = \frac{9}{2} \sqrt{9 - \frac{9}{4}} = \frac{9}{2} \sqrt{\frac{27}{4}} = \frac{27}{4} \sqrt{3} \approx 11.69$   
 $A(3) = 0$  So  $x = \frac{3}{2}$  gives MAXIMUM AREA.

⑦  $D^2 = (x-3)^2 + y^2$  and  $\frac{x^2}{25} + \frac{y^2}{16} = 1 \Rightarrow y^2 = 16(1 - \frac{x^2}{25})$   
 So  $D^2 = (x-3)^2 + 16 - \frac{16}{25}x^2 = F(x) \quad -5 \leq x \leq 5$   
 $F'(x) = 2(x-3) - \frac{32}{25}x = \frac{18}{25}x - 6 = 0 \Rightarrow x = 8\frac{1}{3}$  (OUT OF DOMAIN)  
 So MINIMUM DISTANCE AT ENDPOINTS  $\Rightarrow (5, 0)$  clearly the closest.

⑧  $f(x) = 3x^2 + 2x^3 = x^2(3+2x)$  x-intercept at  $x=0$  and  $x=-3/2$   $f(-1) = 1 \quad f(-\frac{1}{2}) = \frac{1}{2}$   
 $f'(x) = 6x + 6x^2 = 6x(1+x)$  CRIT PTS AT  $x=0, x=-1$   
 $f''(x) = 6 + 12x = 6(1+2x)$   $x = -\frac{1}{2}$  possible P.O.I.  

BELOW	+	ABOVE	+	ABOVE
	-3/2		0	
INCREASE		DECREASE		INCREASE
	-1		0	
CONC. DOWN		CONC. UP		
	-1/2			

⑨  $a(t) = -t^2 \quad x(0) = 0 \quad S_{max} = 4 \Rightarrow v(t) = -\frac{t^3}{3} + C \Rightarrow v(t) = v_0 - \frac{1}{3}t^3$   
 $v(0) = C = v_0$   
 So  $s(t) = v_0 t - \frac{1}{12}t^4 + C_1$   $s(t) = v_0 t - \frac{1}{12}t^4$   
 $s(0) = 0 - 0 + C_1 = 0 \Rightarrow C_1 = 0$   
 MAX DISTANCE when  $v=0 \Rightarrow v_0 - \frac{1}{3}t^3 = 0 \Rightarrow v_0 = \frac{t^3}{3}$   $(3v_0 = t^3)$   
 AT THAT TIME,  $S = 4$ , so  $4 = v_0 t - \frac{1}{12}t^4$   
 SOLVE SIMULTANEOUSLY  $\Rightarrow 4 = \frac{t^4}{3} - \frac{1}{12}t^4 \Rightarrow 48 = 4t^4 - t^4$   
 $\Rightarrow 4t^4 - t^4 = 48 \Rightarrow 3t^4 = 48 \Rightarrow t^4 = 16 \Rightarrow t^2 = \frac{1 \pm \sqrt{1+768}}{3} \Rightarrow t^2 = \frac{1 + \sqrt{769}}{3} = 3.5913 \dots$   
 $\therefore t \approx 1.895 \Rightarrow v_0 = 2.2686 \text{ mi/hr.}$

⑩  $f(x) = x^{2/3} - 2x^{1/3} = x^{1/3}(x^{1/3} - 2) = 0$  when  $x=0, x=8$   
 $f'(x) = \frac{2}{3}x^{-1/3} - \frac{2}{3}x^{-2/3} = \frac{2}{3} \frac{(x^{1/3} - 1)}{x^{2/3}} = 0$  when  $x=1$  vert slope when  $x=0$   
 $f''(x) = -\frac{2}{9}x^{-4/3} + \frac{4}{9}x^{-5/3} = \frac{2}{9} \frac{(-x^{1/3} + 2)}{x^{5/3}} = 0$  when  $x=8$ , possible POI.  
 $x=0$  ALSO possible POI

ABOVE + +      BELOW - -      ABOVE + +  
 DECR      DECR      INCR  
 CONCA DOWN      CONCA UP      CONCA DOWN

$f(1) = -1$

⑪ Area  $A = (2x)y = 2x(12-4x) = 24x - 8x^2 = A(x)$   
 $0 \leq x \leq 3$        $A'(x) = 24 - 16x = 0 \Rightarrow x = \frac{3}{2}$   
 $A(0) = A(3) = 0$        $\therefore$  MAX AREA when  $x = \frac{3}{2}$   
 $A(\frac{3}{2}) = 36 - 8(\frac{9}{4}) = 36 - 18 = 18 = A_{max}$   
 $A_{max} = 18 \text{ in}^2$

⑫  $x_0 = .5 \rightarrow x_1 = -.5 \rightarrow x_2 = .5 \rightarrow x_3 = -.5$  etc.  
 $.5 \rightarrow -.5 \rightarrow +.5 \rightarrow -.5 \rightarrow \dots$   
 CYCLES BACK AND FORTH. BAD INITIAL GUESS.  
 Better initial guesses give roots at  $x = \pm 1.36676 \dots$

⑬ Let  $x = \#$  doctors hired       $10000x + 4000y = 100000 \Rightarrow 5x + 2y = 50$   
 $y = \#$  nurses hired       $N = Kxy = Kx(\frac{50-5x}{2}) = K(25x - \frac{5}{2}x^2)$   
 $0 \leq x \leq 10$        $N'(x) = K(25 - 5x) = 0 \Rightarrow x = 5$   
 $N(0) = 0$        $N(10) = 0$        $N(5) = K(125 - \frac{125}{2}) = 125K/2$   
 $x = 5 \text{ DOCTORS}$  AND  $y = \frac{25}{2} = 12.5 \text{ NURSES}$ .

⑭  $V = \pi r^2 h = 40$       Surf Area  $A = 2\pi r^2 + 2\pi r h$   
 $h = \frac{40}{\pi r^2}$        $= 2\pi r^2 + 2\pi r(\frac{40}{\pi r^2}) = 2\pi r^2 + \frac{80}{r} = A(r)$   
 $A'(r) = 4\pi r - \frac{80}{r^2} = 0 \Rightarrow 4\pi r = \frac{80}{r^2} \Rightarrow r^3 = \frac{20}{\pi}$   
 $\therefore r = \sqrt[3]{\frac{20}{\pi}} = 1.853 \text{ cm}$   
 $h = \frac{40}{\pi r^2} = 3.707 \text{ cm}$   
 Note  $\frac{h}{r} = \frac{40}{\pi r^2} \cdot \frac{1}{r} = \frac{40}{\pi r^3} = \frac{40}{\pi \cdot \frac{20}{\pi}} = 2$   
 $\Rightarrow$  (height = twice the radius)

15) Let  $x = \text{no. of } 5\% \text{ reductions}$   
 price =  $100000(1 - .05x)$   
 people =  $60 + 10x$

$$\Rightarrow \text{Revenue} = [100000(1 - .05x)][60 + 10x]$$

$$= (100000 - 50000x)(60 + 10x)$$

$$\Rightarrow R(x) = 6000000 + 700000x - 50000x^2$$

$$R'(x) = 700000 - 100000x = 0 \Rightarrow x = 7 \Rightarrow \underline{35\% \text{ reduction}}$$

However, we must also consider the capacity of the plane!

$x = 7 \Rightarrow \text{people} = 60 + 70 = 130 \Rightarrow \text{overcapacity}$

$x = 6 \Rightarrow 120 \text{ people is endpoint}$       $R(0) = 6 \text{ million FF}$      **MAXIMUM**  
 $R(6) = 8.4 \text{ million FF} \leftarrow \text{REVENUE}$

16)  $v(t) = t^2 = \frac{dx}{dt}$       $x(t) = \frac{1}{3}t^3 + C$   
 $x(2) - x(1) = (\frac{8}{3} + C) - (\frac{1}{3} + C) = \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3}}$

17)  $a(t) = -5 - 4t \Rightarrow v(t) = -5t - 2t^2 + C$       $x(t) = 25t - \frac{5}{2}t^2 - \frac{2}{3}t^3 + C$   
 $v(0) = 25 \text{ ft/sec}$       $v(0) = 0 + C = 25 \Rightarrow C = 25$       $x(0) = C_1 = 0$   
 $x(0) = 0 \text{ ft}$      **SO**  $v(t) = 25 - 5t - 2t^2$      **SO**  $x(t) = 25t - \frac{5}{2}t^2 - \frac{2}{3}t^3$

18) BAGEL NOT MOVING when  $v = 0 \Rightarrow 2t^2 + 5t - 25 = 0$   
 $t = \frac{-5 \pm \sqrt{25 + 300}}{4} = \frac{-5 \pm 15}{4} \Rightarrow \boxed{t = \frac{5}{2} \text{ sec}}$

$x(0) = 0$   
 $x(5/2) = \frac{125}{2} - \frac{5}{2} \cdot \frac{25}{4} - \frac{2}{3} \cdot \frac{125}{8} = \frac{875}{24} = 36 \frac{11}{24} \text{ feet (outside } [0, 2])$   
 $x(2) = 50 - 10 - \frac{16}{3} = \boxed{34 \frac{2}{3} \text{ feet} = \text{DISTANCE TRAVELLED}}$

18) a)  $f'(x) = \frac{1}{1 + (\frac{x-1}{x+1})^2} \cdot \frac{(x+1) - (x-1)}{(x+1)^2} = \frac{2}{(x+1)^2 + (x-1)^2} = \frac{2}{x^2 + 2x + 1 + x^2 - 2x + 1} = \frac{1}{x^2 + 1}$

Note:  $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{x^2 + 1}$  also, so  $\tan^{-1}(\frac{x-1}{x+1}) - \tan^{-1}(x) = \text{constant}$

b)  $h'(x) = \frac{(x^3 + 2) \left[ x^2 \left( \frac{3}{3x+4} \right) + 2x \ln(3x+4) \right] - [x^2 \ln(3x+4)](3x^2)}{(x^3 + 2)^2}$      [Note:  $\ln(1) = 0$ ]

$h'(-1) = \frac{(1) [1 \cdot 3 - 2 \ln(1)] - [1 \cdot \ln(1)](3)}{1} = \boxed{3}$

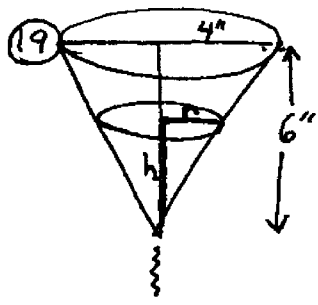
c)  $g'(t) = 3\pi t^2 + (3^{\pi t} \ln 3) \cdot \pi + \pi^3$

d)  $e^{y^2 + y} = xy + 1$      DIFF. IMPLICITLY

$(e^{y^2 + y})(2y + 1) \frac{dy}{dx} = x \cdot \frac{dy}{dx} + y$

plug in  $x=0, y=-1 \Rightarrow -\frac{dy}{dx} = -1 \Rightarrow \boxed{\frac{dy}{dx} = 1}$

(5)



SIMILAR TRIANGLES  $\frac{r}{h} = \frac{4}{6} \Rightarrow 6r = 4h \Rightarrow h = \frac{3}{2}r$

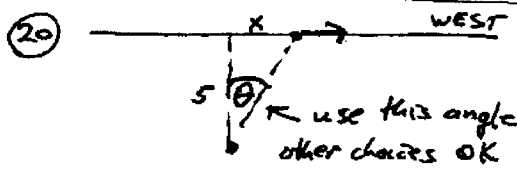
$$V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi r^2 \left(\frac{3}{2}r\right) = \frac{\pi}{2} r^3 = V$$

Take derivative w.r.t. time  $t \Rightarrow \frac{3\pi}{2} r^2 \frac{dr}{dt} = \frac{dV}{dt}$

We know that  $\frac{dV}{dt} = -2 \text{ in}^3/\text{min}$ . When  $h = 3 \text{ in}$ ,  $r = 2 \text{ in}$ .

$$\text{So } \frac{3\pi}{2} \cdot 4 \frac{dr}{dt} = -2 \Rightarrow 6\pi \frac{dr}{dt} = -2 \Rightarrow \boxed{\frac{dr}{dt} = -\frac{1}{3\pi} \text{ in/sec}}$$

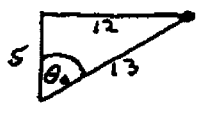
So  $\frac{dh}{dt} = \frac{3}{2} \frac{dr}{dt} = -\frac{1}{2\pi} \text{ in/sec}$



$\frac{dx}{dt} = 507 \text{ mi/hr}$

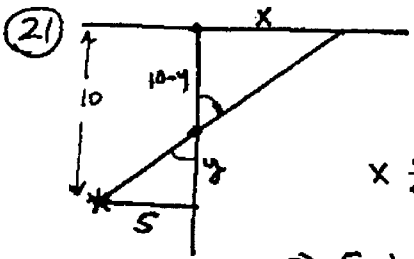
$$\frac{x}{5} = \tan \theta \Rightarrow \frac{1}{5} \frac{dx}{dt} = \sec^2 \theta \frac{d\theta}{dt}$$

When  $x = 12 \text{ mi}$ ,  $\sec \theta = \frac{13}{5}$



$$\text{So } \frac{1}{5}(507) = \left(\frac{13}{5}\right)^2 \frac{d\theta}{dt} = \frac{169}{25} \frac{d\theta}{dt}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{507 \cdot 25}{5 \cdot 169} = 5 \cdot 3 = \boxed{15 \text{ rad/hr}}$$



Similar Triangles  $\frac{x}{10-y} = \frac{5}{y}$

$$\Rightarrow xy = 50 - 5y$$

$\frac{dy}{dt} = 1 \text{ m/s}$

$$x \frac{dy}{dt} + y \frac{dx}{dt} = -5 \frac{dy}{dt}$$

PLUG IN  $\frac{dy}{dt} = 1 \text{ m/s}$

$y = 5, x = 5$

$$\Rightarrow 5 \cdot 1 + 5 \cdot \frac{dx}{dt} = -5 \cdot 1$$

$$\Rightarrow 5 \frac{dx}{dt} = -10 \Rightarrow \frac{dx}{dt} = -2 \text{ m/s}$$

(2 m/s to left)



$$V = \pi r^2 h \quad \frac{dV}{dt} = -5 \text{ ft}^3/\text{hr}$$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt}$$

$r = 500 \text{ ft}, h = .01 \text{ ft}, \frac{dh}{dt} = -.001 \text{ ft/hr}$

$$\Rightarrow -5 = \pi(250000)(-.001) + 2\pi(500)(.01) \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{250\pi - 5}{10\pi} \text{ ft/hr}$$

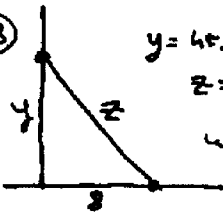
(RADIUS INCREASING)

$$V = A \cdot h \quad \frac{dV}{dt} = A \frac{dh}{dt} + h \frac{dA}{dt} \Rightarrow -5 = \pi(250000)(-.001) + (.01) \frac{dA}{dt}$$

$$\text{So } \frac{dA}{dt} = \frac{250\pi - 5}{.01} = (25000\pi - 500) \text{ ft}^2/\text{hr} \quad \text{INCREASING}$$

Note! Could also use  $A = \pi r^2 \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 1000\pi \frac{dr}{dt}$  to get this.

(23)



$y = \text{ht. of monkey above ground}$   
 $z = \text{length of rope}$   
 weight of monkey is irrelevant

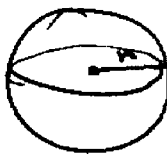
$$6^2 + y^2 = z^2$$

$$2y \frac{dy}{dt} = 2z \frac{dz}{dt}$$

$$12 \cdot 4 = 20 \frac{dz}{dt} \Rightarrow \frac{dz}{dt} = \boxed{2.4 \text{ ft/sec}}$$

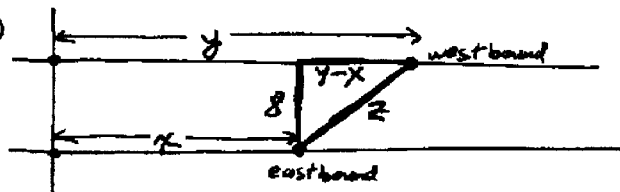
$y = 6' \Rightarrow z = 10'$   
 $\frac{dy}{dt} = 4 \text{ ft/sec}$

(24)



$V = \frac{4}{3} \pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$   
 If  $\frac{dr}{dt} = 2 \text{ in/min}$ ,  $r = 5 \text{ in}$ , then  $\frac{dV}{dt} = 4\pi \cdot 25 \cdot 2 = \boxed{200\pi \text{ in}^3/\text{min}}$

(25)



$\frac{dy}{dt} = -20 \text{ naut mi/hr}$   
 $\frac{dx}{dt} = +20 \text{ naut mi/hr}$

$$8^2 + (y-x)^2 = z^2$$

$$64 + (y-x)^2 = z^2$$

Differentiate w.r.t time  $\Rightarrow 2(y-x) \left( \frac{dy}{dt} - \frac{dx}{dt} \right) = 2z \frac{dz}{dt}$

When ships are 10 naut mi apart,  $z = 10 \Rightarrow (y-x) = 6 \text{ mi}$

So  $2(6)(-20 - 20) = 2 \cdot 10 \frac{dz}{dt}$

$\Rightarrow 12(-40) = 20 \frac{dz}{dt}$

$\Rightarrow \frac{dz}{dt} = -\frac{480}{20} = \boxed{-24 \text{ naut mi/hr}}$