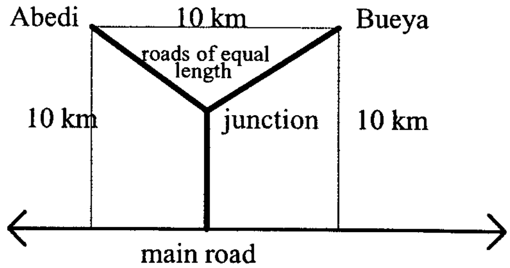
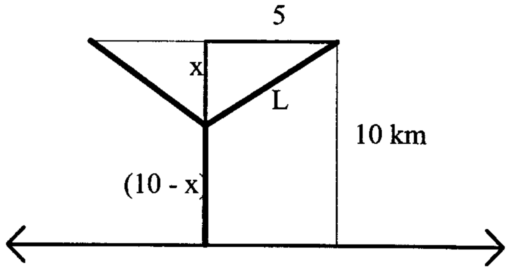
Solutions to Math 1a Exam #2: Tues, Nov 23, 1999

(1) Consider the two small Zairian towns of Abedi and Bueya, which currently cannot be reached by road. The main road from Kinshasa to Matadi passes 10 km south of the two towns.

The government plans to build a road system connecting the two towns to the main road , as shown in the diagram. How far from the main road should the junction point be chosen to minimize the total length of the new roads?



Soln: There are many good choices for labeling the diagram and defining variables. Here's one way:



Here, the total length of roads is 2L + (10 - x) = $2\sqrt{x^2+25}+10-x$ = $f(x)$, where $0 \le x \le 10$.

The endpoints give $f(0) = 20$ and $f(10) = 2\sqrt{125} = 10\sqrt{5} \cong 22.4$.

For critical points, $f'(x) = \dfrac{2x}{\sqrt{x^2+25}} - 1 = 0$ can be solved to give $x^2 + 25 = 4x^2$, so $x^2 = \frac{25}{3}$ and $x = \dfrac{5}{\sqrt{3}}$.

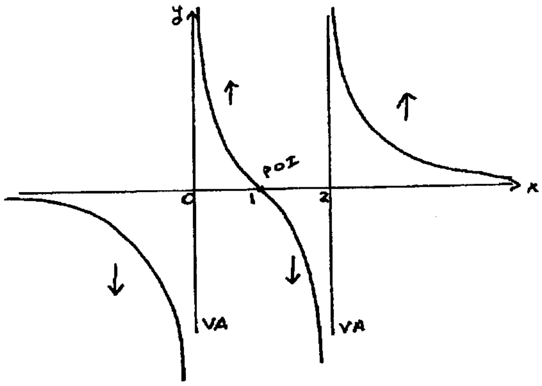
This yields a total length of $2\sqrt{\frac{25}{3}+25}+10-\dfrac{5}{\sqrt{3}} = \dfrac{20}{\sqrt{3}}+10-\dfrac{5}{\sqrt{3}} = \dfrac{15}{\sqrt{3}}+10 = 5\sqrt{3}+10 \cong 18.66$, so this is the minimum. You can also verify that this is the minimum length by the second derivative test.

(2) A creature moves along the x-axis in such a way that its acceleration is given by $a(t) = 4 - 12t$. We also know that when $t = 1$ it is at the position $x = 3$ and has zero velocity at that moment. Where will the creature be when $t = 2$?

Soln: Start with $a(t) = 4 - 12t$. Antidifferentiation gives $v(t) = C_1 + 4t - 6t^2$.
Plug in $t = 1$, $v = 0$ to get that $C_1 = 2$. So $v(t) = 2 + 4t - 6t^2$.
Antidifferentiate again to get $x(t) = C_2 + 2t + 2t^2 - 2t^3$. Plug in $t = 1$, $x = 3$ to get that $C_2 = 1$.
So $x(t) = 1 + 2t + 2t^2 - 2t^3$. Finally, calculate that $x(2) = 1 + 4 + 8 - 16 = -3$.

(3) A function $f(x)$ and its first and second derivatives are given below.

Soln: One x-intercept at $x = 1$, no critical points, vertical asymptotes at $x = 0$ and $x = 2$, a point of inflection at $x = 1$, and the horizontal asymptote $y = 0$.

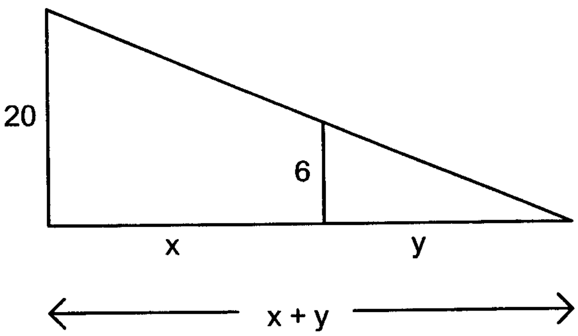


Sketch for Problem 3

(4) A manufacturing plant has a capacity of 25 articles per week. Experience has shown that n articles can be sold at a price of p dollars each where p = 110 - 2n and the cost of producing n articles is known to be 600 + 20n + n^2 dollars. How many articles should be made each week to give the largest profit?

Soln: The profit function is given by $P(n) = n(110 - 2n) - (600 + 20n + n^2) = -3n^2 + 90n - 600$ where $0 \leq n \leq 25$. Differentiate and set equal to zero to find the critical point n = 15. Since the 2nd derivative is always negative, this must give the maximum profit, i.e. 15 articles manufactured.

(5) A man 6 feet tall walks away from a source of light 20 feet above the ground at a speed of 3 feet per second. At what rate is the shadow of his head moving along the ground?



Soln: We are given that $\frac{dx}{dt} = 3$ ft/sec and we'd like to calculate how fast the tip of the shadow is moving along the ground, i.e. how fast the distance from a fixed point on the ground to the tip of the shadow is changing. If we use the point directly below the light, then we're looking for $\frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt}$. Looking at similar triangles, we have that $\frac{x+y}{20} = \frac{y}{6}$, so $6x + 6y = 20y$ and, therefore $6x = 14y$. This gives us that $6\frac{dx}{dt} = 14\frac{dy}{dt}$, so $\frac{dy}{dt} = \frac{6}{14}\frac{dx}{dt} = \frac{3}{7}(3) = \frac{9}{7}$ ft/sec. Consequently $\frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt} = 3 + \frac{9}{7} = \frac{30}{7}$ ft/sec.
A common error on this problem was to find only the rate at which the length of the shadow was changing. There were also some more direct ways to use similar triangles to solve the problem.

(6) Solutions:

(a) Two applications of L'Hopital's Rule give $\lim_{x\to 0} \frac{\cos x - 1}{x^3} = \lim_{x\to 0} \frac{-\sin x}{3x^2} = \lim_{x\to 0} \frac{-\cos x}{6x}$, which is undefined.

(b) Again, two applications of L'Hopital's Rule give $\lim_{x\to +\infty} \frac{e^{2x}}{3x^2+1} = \lim_{x\to +\infty} \frac{2e^{2x}}{6x} = \lim_{x\to +\infty} \frac{4e^{2x}}{6} \to +\infty$

(c) One application of L'Hopital's Rule and some algebra give:

$$\lim_{x\to 1} \frac{x^2 \ln x}{x^3 - 1} = \lim_{x\to 1} \frac{x^2 \frac{1}{x} + 2x \ln x}{3x^2} = \lim_{x\to 1} \frac{x(1 + 2\ln x)}{3x^2} = \lim_{x\to 1} \frac{(1 + 2\ln x)}{3x} = \frac{1}{3}.$$

(7) Find the point on the parabola $y = x^2$ that is closest to the point (4, 1/2).

Soln: The square of the distance from this point to any point (x, y) on the parabola is given by $(x - 4)^2 + (y - 1/2)^2 = (x - 4)^2 + (x^2 - 1/2)^2 = x^4 - 8x + \frac{65}{4} = f(x)$.

Differentiate to find the critical point: $4x^3 - 8 = 0 \Rightarrow x = \sqrt[3]{2}$, $y = \left(\sqrt[3]{2}\right)^2$. Observation or the 2nd derivative test can be used to verify that this is, indeed, the closest point.