Name: ______________________ ID#: ______________

Solutions to Midterm II

Math 1a
Introduction to Calculus

8 December 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

Please check your section:

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Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.

—*Handbook for Students*

1. (12 Points) *Find the following derivatives. You may use any "standard" facts about differentiation.*

(i) $\dfrac{d}{du}(u^3+2)^{17}$

Solution. By the chain rule,

$$\frac{d}{du}(u^3+2)^{17} = 17(u^3+2)^{16}(3u^2) = 51\,u^2\left(2+u^3\right)^{16}$$

□

(ii) $\dfrac{d}{dx}\left(e^{\sin(x)+19}\right)$

Solution. Again, this is by the chain rule:

$$\frac{d}{dx}\left(e^{\sin(x)+19}\right) = e^{\sin(x)+19}\cos(x).$$

□

(iii) $\dfrac{d}{ds}\ln(\tan s)$

Solution.

$$\frac{d}{dx}\ln(\tan s) = \frac{1}{\tan s}\frac{d}{ds}\tan s = \frac{1}{\tan s}\sec^2 x = \csc s\ \sec s.$$

□

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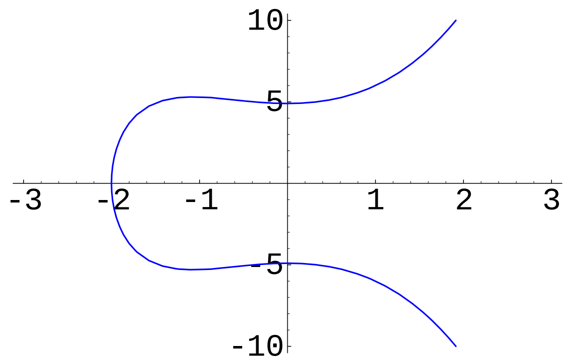
2. (10 Points) *Consider the equation*

$$y^2 = (x^2 + 3)(x^3 + 8)$$

This equation defines a curve in the plane.



(a) *Calculate* $\frac{dy}{dx}$.

Solution. If we differentiate the relation with respect to x, we get

$$2\,y\,\frac{dy}{dx} = 3\,x^2\,\left(3 + x^2\right) + 2\,x\,\left(8 + x^3\right)$$

Solving this for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = \frac{16\,x + 9\,x^2 + 5\,x^4}{2\,y}.$$

□

(b) *The graph of the curve has two tangent lines at* $x = 1$. *Find the point at which these two lines intersect.*

Solution. The two points on the curve with $x = 1$ are $(1, 6)$ and $(1, -6)$. The slopes of the lines tangent to the curve at these points are $\frac{dy}{dx}$ evaluated at them. One of the lines has slope

$$\left.\frac{dy}{dx}\right|_{(1,6)} = \frac{30}{12} = \frac{5}{2},$$

and therefore its equation is

$$y - 6 = \tfrac{5}{2}(x - 1).$$

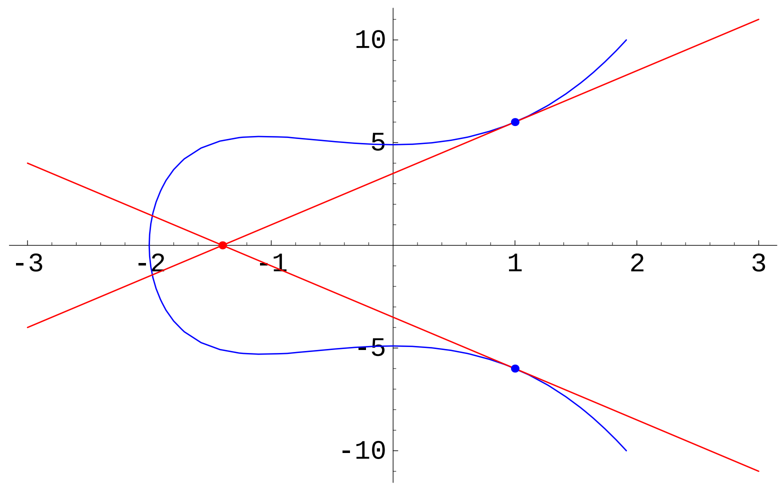
The other line is

$$y + 6 = -\tfrac{5}{2}(x - 1).$$

Where the two lines intersect, both equations hold and we have

$$\begin{aligned} \frac{5}{2}(x-1) + 6 &= \frac{5}{2}(x-1) - 6 \\ 5x + 7 &= -5x - 7 \\ x &= \frac{7}{5}. \end{aligned}$$

It follows that $y = 0$ at this intersection.



□

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3. (10 Points) *In this question, you will use logarithmic differentiation to find the derivative of a product of functions without using the product rule. Assume that $f(x)$ and $g(x)$ are always positive.*

(a) *Take the natural log of both sides of the equation $y = f(x)g(x)$ and use properties of logarithms to rewrite the right-hand side.*

Solution.

$$\ln(y) = \ln\left(f(x)g(x)\right) = \ln(f(x)) + \ln(g(x))$$

□

(b) *Differentiate your answer to part (a) with respect to x.*

Solution.

$$\frac{1}{y}\frac{dy}{dx} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}$$

□

(c) *Use algebra to write $\frac{dy}{dx}$ as a function of x only. (y cannot appear in your formula for $\frac{dy}{dx}$.)*

Solution. We just multiply both sides by $y = f(x)g(x)$:

$$\frac{dy}{dx} = f(x)g(x)\left(\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right) = g(x)f'(x) + f(x)g'(x).$$

□

(d) *Is this the same answer you would have obtained by applying the product rule directly?*

Solution. Yes! □

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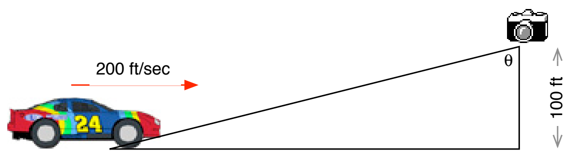
4. (10 Points) *You are videotaping a race from a stand 100 feet from the track, following a car that is moving 200 feet per second. How fast will your camera angle θ be changing two seconds before the car reaches the point on the track that is directly in front of you? Assume the track is a straight line.*



Solution. Let x be the distance from the car to the point directly in front of the camera. Then

$$\tan\theta = \frac{x}{100}.$$

Differentiating with respect to time, we have

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{100}\frac{dx}{dt}. \tag{4.1}$$

Two seconds before the car is directly ahead, it is 400 feet away. The right triangle can be filled in with horizontal leg 400 and hypotenuse $100\sqrt{17}$. Then we can substitute into (4.1):

$$17\frac{d\theta}{dt} = \frac{1}{100}(-200)$$
$$\implies \frac{d\theta}{dt} = -\frac{2}{17}\frac{\text{feet}}{\text{sec}}$$

□

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5. (20 Points) *Let* $f(x) = \dfrac{1}{x-1} + \dfrac{1}{2(x-1)^2}$.

(a) State the domain of f.

Solution. The only problem points for f are when the denominators become zero, which happens only when $x = 1$. So the domain of f is $(-\infty, 1) \cup (1, \infty)$. □

(b) Find zeroes of f *and the intervals where it is positive or negative.*

Solution. We can add the two fractions together to get a factored form of f:

$$y = \frac{1}{x-1} + \frac{1}{2(x-1)^2} = \frac{2(x-1)+1}{2(x-1)^2} = \frac{2x-1}{2(x-1)^2}.$$

The denominator is always positive for $x \neq 1$, and so the sign is determined by the numerator, $2x - 1$. Hence $y < 0$ on $(-\infty, \frac{1}{2})$, and $y > 0$ on $(\frac{1}{2}, \infty)$. $y = 0$ at $x = \frac{1}{2}$.

We can summarize this information in a sign chart:

	$x < \frac{1}{2}$	$\frac{1}{2} < x < 1$	$x > 1$
$2x - 1$	$-$	$+$	$+$
$2(x-1)^2$	$+$	$+$	$+$
$f(x)$	$-$	$+$	$+$

□

(c) Find all vertical and horizontal asymptotes of f.

Solution. The only possibility of a vertical asymptote is at $x = 1$. For x near 1, $2x - 1$ is near 1, and $2(x-1)^2$ is very small and positive. Thus y is large and positive. So

$$\lim_{x \to 1} \frac{2x-1}{2(x-1)^2} = \infty,$$

giving a vertical asymptote of $x = 1$.

To find the horizontal asymptotes, we note that

$$\lim_{x \to \infty} \frac{2x-1}{2(x-1)^2} = 0,$$

since the degree of the numerator is greater than the degree of the denominator. The same is true of $\lim_{x \to -\infty} f(x)$. So $y = 0$ is the only horizontal asymptote of f. □

(d) *Find the intervals of increase or decrease, and any local maxima or minima.*

Solution. We will need to know the derivative of f, which we choose to write in a slightly different form:

$$\begin{aligned} y &= (x-1)^{-1} + \tfrac{1}{2}(x-1)^{-2} \\ \frac{dy}{dx} &= -(x-1)^{-2} - (x-1)^{-3} \\ &= (-1)(x-1)^{-3}(x-1) + 1 = -x(x-1)^{-3}. \end{aligned}$$

We can see that $f'(x) = 0$ exactly when $x = 0$. We also need to check f' on either side of the 1, where the derivative is not defined. We can make a sign chart:

	$x < 0$	$0 < x < 1$	$x > 1$
-1	$-$	$-$	$-$
x	$-$	$+$	$+$
$(x-1)^{-3}$	$-$	$-$	$+$
$f'(x)$	$-$	$+$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$

So f is decreasing on $(-\infty, 0)$ and $(1, \infty)$, and decreasing on $(0, 1)$. The point $(0, -\frac{1}{2})$ is a local minimum. □

(e) *Find the intervals of concavity and the inflection points.*

Solution. We need the second derivative of f:

$$\begin{aligned} y' &= -(x-1)^{-2} - (x-1)^{-3} \\ y'' &= 2(x-1)^{-3} + 3(x-1)^{-4} \\ &= (x-1)^{-4}(2(x-1)+3) = \frac{2x+1}{(x+1)^4}. \end{aligned}$$

We can see that $f''(x) = 0$ exactly when $x = -\frac{1}{2}$. As before, we will also need to check on either side of $x = 1$. So we make another sign chart:

	$x < -\frac{1}{2}$	$-\frac{1}{2} < x < 1$	$x > 1$
$2x+1$	$-$	$+$	$+$
$(x-1)^{-4}$	$+$	$+$	$+$
$f''(x)$	$-$	$+$	$+$
$f(x)$	$\frown$	$\smile$	$\smile$

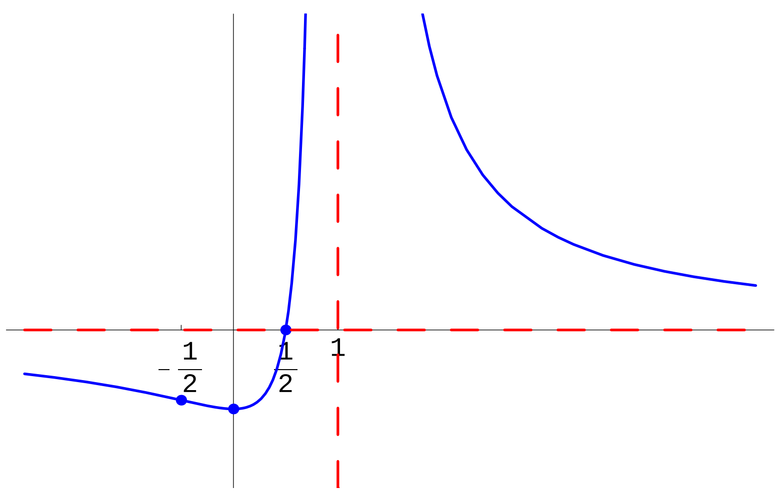
So f is concave up on $(-\frac{1}{2}, 1)$ and $(1, \infty)$, and concave down on $(-\infty, -\frac{1}{2})$. There is a point of inflection at $(-\frac{1}{2}, -\frac{4}{9})$. □

(f) Sketch the graph of f. Find the global minimum and maximum, if they exist.

Solution. We can put all of this information together.

- If $x < -\frac{1}{2}$, f is negative, decreasing, and concave down. Moreover, $\lim_{x\to-\infty} f(x) = 0$.
- The point $(-\frac{1}{2}, -\frac{4}{9})$ is a point of inflection.
- If $-\frac{1}{2} < x < 0$, f is negative, decreasing, and concave up.
- The point $(0, -\frac{1}{2})$ is a local minimum.
- If $0 < x < \frac{1}{2}$, f i s negative, increasing, and concave up.
- The function crosses 0 at $x = \frac{1}{2}$.
- If $\frac{1}{2} < x < 1$, f is positive, increasing, and concave up.
- $x = 1$ is a horizontal asymptote, with f increasing without bound on either side.
- If $x > 1$, f is positive, decreasing, and concave up.
- If $0 < x < \frac{1}{2}$, f is negative, increasing, and concave up. Moreover, $\lim_{x\to\infty} f(x) = 0$.

The graph is below.



There is no global maximum because of the infinite limit at $x = 1$. The global minimum is the local minimum at $(0, -\frac{1}{2})$. □

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6. (12 Points) *Find each of the following limits, with justification. Indicate each point at which you apply L'Hôpital's rule.*

(i) $\lim_{x\to 0} \frac{\arcsin(x)}{\ln(x+1)}$

Reminder. $\arcsin(x)$ is written in the book as $\sin^{-1}(x)$, but this is *not* the same thing as $\frac{1}{\sin(x)}$.

Solution. This is a limit of the form $\frac{0}{0}$. So we can find it using L'Hôpital's Rule:

$$\begin{aligned}\lim_{x\to 0} \frac{\arcsin(x)}{\ln(x+1)} &\overset{\text{H}}{=} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{1}{x+1}} \\ &= \lim_{x\to 0} \frac{x+1}{\sqrt{1-x^2}} \\ &= \frac{0+1}{\sqrt{1-0^2}} = 1.\end{aligned}$$

□

(ii) $\lim_{x\to 0} \frac{\sin(x)}{x+1}$

Solution. This was a "red herring" where L'H'ôpital's Rule does not apply. Indeed, the numerator approaches 0 but the denominator approaches 1. So the limit is 0. □

(iii) $\lim_{x\to 0^+} \sin(x)^{\sin(x)}$

Solution. This limit is of the form 0^0, and indeterminate power, so we can take logarithms and apply L'Hôpital's Rule. We have:

$$\begin{aligned}\ln \lim_{x\to 0^+} \sin(x)^{\sin(x)} &= \lim_{x\to 0^+} \ln \sin(x)^{\sin(x)} \\ &= \lim_{x\to 0^+} \sin(x) \ln(\sin(x)) \\ &= \lim_{x\to 0^+} \frac{\ln(\sin(x))}{\csc(x)} \\ &\overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\csc(x)\cos(x)}{-\csc(x)\cot(x)} \\ &= \lim_{x\to 0^+} -\sin x = 0.\end{aligned}$$

Therefore

$$\lim_{x\to 0^+} \sin(x)^{\sin(x)} = e^0 = 1.$$

□

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7. (6 Points) *Use linear approximation to find the approximate value of* $\ln(1.02)$.

Solution. The linear approximation to f at a is given by

$$L(x) = f(a) + f'(a)(x - a),$$

If $f = \ln$, then $f(1) = 0$ and $f'(1) = 1$. So the linear approximation is $x - 1$. So

$$\ln(1.02) \approx .02$$

□

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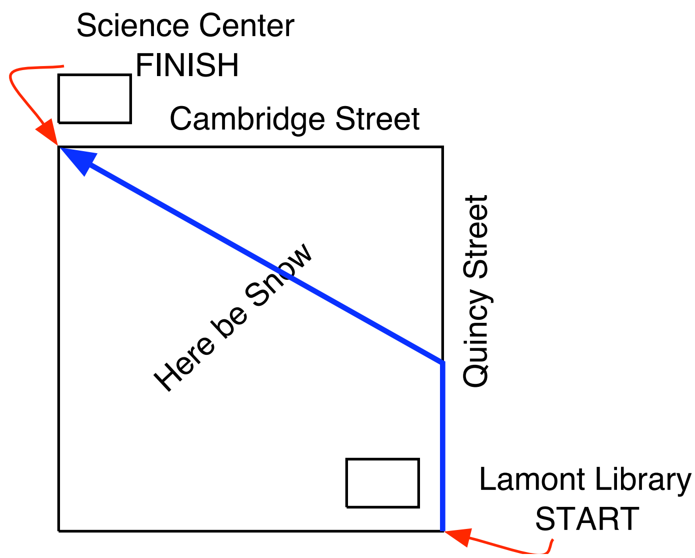
8. (10 Points) *Ferdbert Freshman wants to get from his study carrell at Lamont Library to a problem session at the Science Center. The most direct path would be diagonally across Harvard Yard. Unfortunately, a foot of snow has just fallen, and Ferdbert walks only $\frac{1}{\sqrt{5}}$ as fast on snow as on pavement. The sidewalks around the Yard (along Quincy Street and Cambridge Street) are plowed. Ferdbert decides to take a mixed approach, walking North along Quincy Street for a certain distance, then cutting diagonally across the rest of the yard to the Science Center. What is the minimum amount of time that Ferdbert can take to make his trip?*



We'll assume that Harvard Yard is a square of side length 200 meters, and Ferdbert's normal walking speed is 10 kilometers per hour. Also, ignore the fact that there is a wall around Harvard Yard and there are only finitely many places to physically enter.

Solution. It's often preferable to leave constants of the problem as letters. The algebra is a little bit more manageable because you don't have to manage all your 200^2's and $10/\sqrt{5}$'s. So let the side length of the yard be w and Ferbert's speed on the street be v.

It's important to remember what is being minimized: *total time.* If speed is constant, remember the formula

$$\text{distance} = \text{speed} \times \text{time}$$
$$\implies \text{time} = \frac{\text{distance}}{\text{speed}}.$$

Now Ferdbert's speed is not constant, so we have to split up his journey into the portions where it is:

$$\text{total time} = \frac{\text{distance on street}}{\text{speed on street}} + \frac{\text{distance on snow}}{\text{speed on snow}}.$$

If x is the distance walked along Quincy Street, the distance across the Yard is $\sqrt{w^2 + (w-x)^2}$ by the Pythagorean Theorem. Hence

$$t = \frac{x}{v} + \frac{\sqrt{w^2 + (w-x)^2}}{v/\sqrt{5}}.$$

This is the expression we want to optimize, and it's all in terms of the single variable x, which varies between $x = 0$ (walk diagonally across from Lamont to the Science Center) and $x = w$ (walk all the way around the yard). So we can use the Closed Interval Method. We have

$$\begin{aligned} t(0) &= \frac{\sqrt{2}w}{v/\sqrt{5}} = \frac{\sqrt{10}w}{v} \\ t(w) &= \frac{w}{v} + \frac{w}{v/\sqrt{5}} = \frac{w}{v}(1+\sqrt{5}). \end{aligned}$$

(Actually, the formula we have derived for t does not quite apply to the current situation. For we have assumed Cambridge Street is *plowed* and so the actual time to go all the way around the Yard is $\frac{w}{v} + \frac{w}{v} = \frac{2w}{v}$. But that makes our t function discontinuous, and the optimization a little bit more subtle. No student or teacher, not even the course head, caught this subtely, so let's pretend that the problem was written with Cambridge Street not plowed. Problems were graded correctly either way.)

Now for the hard part, which is finding any critical points. We have

$$\frac{dt}{dx} = \frac{1}{v}\left[1 - \frac{\sqrt{5}(w-x)}{\sqrt{w^2+(w-x)^2}}\right],$$

so $\frac{dt}{dx} = 0$ when

$$\begin{aligned} \frac{\sqrt{5}(w-x)}{\sqrt{w^2+(w-x)^2}} &= 1 \\ \frac{5(w-x)^2}{w^2+(w-x)^2} &= 1 \\ 5(w-x)^2 &= w^2 + (w-x)^2 \\ 4(w-x)^2 &= w^2 \\ (w-x)^2 &= \frac{w^2}{4} \\ w - x &= \pm\frac{w}{2} \end{aligned}$$

This leaves us with $x = \frac{w}{2}$ (we throw out the solution $x = \frac{3w}{2}$ since it's not in the interval we want).

Now

$$\begin{aligned} t(w/2) &= \frac{w}{2v} + \frac{\sqrt{5}}{v}\sqrt{w^2 + \frac{w^2}{4}} \\ &= \frac{w}{2v} + \frac{\sqrt{5}}{v}\sqrt{\frac{5w^2}{4}} \\ &= \frac{w}{2v} + \frac{5w}{2v} \\ &= \frac{6w}{2v} = \frac{3w}{v}. \end{aligned}$$

We have to find the smallest of $t(0) = \frac{\sqrt{10}}{v}$, $t(w/2) = \frac{3w}{v}$, and $t(w) = (1 + \sqrt{5})\frac{w}{v}$. Since they all have the same factor of $\frac{w}{v}$, we only have to find the smallest of $\sqrt{10}$, 3, and $1 + \sqrt{5}$.

We know that $\sqrt{10} > 3$ (since $10 > 9$), and similarly, $3 > \sqrt{5} > 2$. So $\sqrt{5}+1$ and $\sqrt{10}$ are both bigger than 3. Thus the middle point $t(w/2)$ is the smallest of the three.

(Back to the subtlety: Remember that if Cambridge Street is plowed, the time to walk all the way around the yard is $2w/v$. That turns out to be smaller than all three of these). ☐

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9. (10 Points) *Sharks and snorks used to coexist peacefully in a far away ocean; at one point in time there were 100 snorks and they reproduced at the rate of* 4 *new snorks every day. A year passed, however, and the sharks could no longer repress the urge to sink their teeth into those juicy snorks. Although the snorks kept reproducing at their normal rate, sharks began eating snorks at a rate of* at most 2.5 *snorks eaten every day, and* at least 2.3 *snorks eaten every day.*

Assume that the snork population is a differentiable function of time. Give upper and lower estimates for the population of snorks t days after the onset of the shark attacks. Use the Mean Value Theorem to justify your assertions.

Solution. Let $S(t)$ denote the number of snorks at time t. Then after a year has passed, the snork population is

$$S(0) = 100 + 4(365) = 1560.$$

For $t \geq 0$, however, we have

$$4 - 2.5 = 1.5 \leq S'(t) \leq 4 - 2.3 = 1.7.$$

Consider the difference quotient $\dfrac{S(t) - S(0)}{t - 0}$. By the Mean Value Theorem, this is equal to $S'(t^*)$ for some t^* with $0 < t^* < t$. We know $1.5 < S'(t^*) < 1.7$, so

$$1.5 \leq \frac{S(t) - S(0)}{t - 0} \leq 1.7.$$

Solving this for $S(t)$ gives

$$1.5t + S(0) \leq S(t) \leq 1.7t + S(0)$$

or,

$$1560 + 1.5t \leq S(t) \leq 1560 + 1.7t.$$

□

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