Definite Integrals and the Fundamental Theorem of Calculus

1. (a) $\int_0^{0.5} p(t)\,dt$

   (b) Note that the change in height of the pine tree after 1 year equals $\int_0^1 p(t)\,dt$, which also equals the area under the curve $y = p(t)$ from $t = 0$ to $t = 1$. Similarly, the change in height of the fir tree after 1 year equals $\int_0^1 f(t)\,dt$, which also equals the area under the curve $y = f(t)$ from $t = 0$ to $t = 1$. Since the area under $y = f(t)$ from $t = 0$ to $t = 1$ is greater than the area under $y = p(t)$ from $t = 0$ to $t = 1$, it follows that the fir tree is taller after 1 year.

   (c) By similar reasoning, the pine tree is taller after 2 years.

   (d) The trees are growing at the same rate when $f(t) = p(t)$, since these functions give the rates of growth for the trees. Thus the trees are growing at the same rate at $t = 0$ and about $t = 1$.

   (e) The trees are the same height when the area under the curve $y = f(t)$ and the area under the curve $y = p(t)$ are equal. This occurs at $t = 0$ and about $t = 1.6$. Note that $t = 1.6$ is just a guess based on an estimation of the areas under the two curves.

2. Note that the integral $\int_2^6 f(x)\,dx$ equals the area between the curve and the x-axis from $x = 2$ to $x = 3$ minus the area between the curve and the x-axis from $x = 3$ to $x = 4$ plus the area between the curve and the x-axis from $x = 4$ to $x = 5$ minus the area between the curve and the x-axis from $x = 5$ to $x = 6$. Given the relative sizes of these areas, this is a positive quantity.

   The integral $\int_2^8 f(x)\,dx$ equals $\int_2^6 f(x)\,dx$ plus the area between the curve and the x-axis from $x = 6$ to $x = 7$ minus the area between the curve and the x-axis from $x = 7$ to $x = 8$. Given the relative sizes of these areas, this is slightly greater than $\int_2^6 f(x)\,dx$.

   Note that the integral $\int_0^1 f(x)\,dx$ is the area between the curve and the x-axis from $x = 0$ to $x = 1$. This is a positive quantity and likely much larger than $\int_2^8 f(x)\,dx$. Thus we have

   $$\int_2^6 f(x)\,dx < \int_2^8 f(x)\,dx < \int_0^1 f(x)\,dx.$$

   The integral $\int_2^2 f(x)\,dx$ equals zero, which is less than the three integrals discussed thus far.

   The integrals $\int_1^2 f(x)\,dx$ and $\int_5^6 f(x)\,dx$ are both negative, and the integral $\int_1^2 f(x)\,dx$ is a larger negative number than the integral $\int_5^6 f(x)\,dx$. Thus we have that

   $$\text{(e)} < \text{(b)} < \text{(c)} < \text{(f)} < \text{(a)} < \text{(d)}.$$

3. (a) As x gets larger, $A(x)$ gets larger on those intervals on which f is positive. Thus, A is increasing on the intervals $(0, 2)$, $(4, 7)$, and $(9, 10)$.

   (b) As x gets larger, $A(x)$ gets smaller on those intervals on which f is negative. Thus, A is decreasing on the intervals $(2, 4)$ and $(7, 9)$.

   (c) From our answers to parts (a) and (b), we can see that A has local maxima at $x = 2$ and $x = 7$ and local minima at $x = 4$ and $x = 9$.

   (d) Note that $A(0) = 0$ but $A(x) > 0$ for all x in $(0, 10]$. Thus A has an absolute minimum at $x = 0$. While the absolute maximum of A on the interval $[0, 7]$ occurs at $x = 7$, it is unclear what the absolute maximum of A on the interval $[0, 10]$ is. If

   $$\left|\int_7^9 f(t)\,dt\right| > \int_9^{10} f(t)\,dt$$

   , then $x = 7$ is the absolute maximum of A over the entire interval. Otherwise, $x = 10$ is.

(e) Consider the interval $(0, 1)$. On this interval A is increasing (because f is positive). Furthermore, it is increasing at an increasing rate, since f is increasing on this interval. Similar reasoning shows that A is concave up when f is increasing, so A is concave up on the intervals $(0, 1)$, $(3, 5.5)$, and $(8, 10)$, approximately.

(f) Likewise, A is concave down when f is decreasing, so A is concave down on the intervals $(1, 3)$ and $(5.5, 8)$, approximately.