L'Hôpital's Rule

1. (a) Multiplying a very small number by a very large number might give you a very small number (if the first term decreases much faster than the second term increases) or a very large number (if the second term increases much faster than the first term decreases).

2. (c) Multiplying a number very close to 1 by a very large number gives a very large number.

3. (c) Multiplying two very large numbers gives a another very large number.

4. (a) Taking a very small number to a very small power might give you a very small number (if the base decreases much faster than the power decreases) or a number very close to 1 (if the power decreases much faster than the base decreases).

5. (b) Taking a very small number to a very large power gives an even smaller number.

6. (a) Taking a number very close to 1 to a very large power might give you a number very close to 1 (if the base approaches 1 much faster than the exponent increases) or a very large number (if the exponent increases much faster than the base approaches 1).

7. (a) Taking a very large number to a very small power might give you a number very close to 1 (if the power decreases much faster than the base increases) of a very large number (if the base increases much faster than the power decreases).

8. (c) Taking a very large number to a very large power gives you a very large number.

9. (a) Consider

$$\lim_{x\to\infty} \frac{e^x}{x^{1,000,000}}.$$

This is a limit of indeterminate form $\frac{\infty}{\infty}$, and so L'Hôpital's Rule gives us

$$\lim_{x\to\infty} \frac{e^x}{x^{1,000,000}} = \lim_{x\to\infty} \frac{\frac{d}{dx}e^x}{\frac{d}{dx}x^{1,000,000}} = \lim_{x\to\infty} \frac{e^x}{1,000,000x^{999,999}}.$$

This is also a limit of indeterminate form $\frac{\infty}{\infty}$, and so L'Hôpital's Rule can again be applied. In fact, one can apply L'Hôpital's Rule $1,000,000$ times to the original limit to get

$$\lim_{x\to\infty} \frac{e^x}{1,000,000!x^0} = \lim_{x\to\infty} \frac{e^x}{1,000,000!}.$$

While $1,000,000!$ is a very large number, for large enough x, e^x is larger. Thus

$$\lim_{x\to\infty} \frac{e^x}{x^{1,000,000}} = \lim_{x\to\infty} \frac{e^x}{1,000,000!} = \infty.$$

This tells us that for very large x-values, the ratio of e^x to $x^{1,000,000}$ is very large, and so $f(x)e^x$ must eventually grow faster than $g(x) = x^{1,000,000}$.

10. (b) An application of L'Hôpital's Rule to $\infty - \infty$ type of limits.

$$\lim_{x\to\infty} [xe^{1/x} - x] = \lim_{x\to\infty} \frac{e^{1/x} - 1}{1/x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(e^{1/x} - 1)}{\frac{d}{dx}1/x} = \lim_{x\to\infty} \frac{e^{1/x}(-x^{-2})}{-x^{-2}} = \lim_{x\to\infty} e^{1/x} = 1.$$