Math 1a Fall 2004
Midterm I Review Solutions

1. Note that the domain of f is $\{x \mid x^4 - 4 > 0\} = (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

   If we substitute $-\sqrt{2}$ into f, we get $-2\sqrt{2}$ in the numerator and 0 in the denominator. This indicates that $x = -\sqrt{2}$ is a vertical asymptote. Since f is negative to the left of $x = -\sqrt{2}$, it follows that

   $$\lim_{x \to -\sqrt{2}^-} f(x) = -\infty.$$

   If we substitute $\sqrt{2}$ into f, we get $2\sqrt{2}$ in the numerator and 0 in the denominator. This indicates that $x = \sqrt{2}$ is a vertical asymptote. Since f is positive to the right of $x = \sqrt{2}$, it follows that

   $$\lim_{x \to \sqrt{2}^+} f(x) = \infty.$$

   Note that

   $$\lim_{x \to -\sqrt{2}^+} f(x) \qquad \text{and} \qquad \lim_{x \to \sqrt{2}^-} f(x)$$

   do not exist since f is not defined to the right of $-\sqrt{2}$ and to the left of $\sqrt{2}$.

   Since

   $$\lim_{x \to \pm\infty} f(x) = 0$$

   (note that the numerator as degree 1 and the denominator has "degree" 2), f has a horizontal asymptote at $y = 0$.

2. Let $g(x) = f(x) - x$. If $g(c) = 0$ for some c, then $0 = g(c) = f(c) - c$ and so $f(c) = c$. We'll use the Intermediate Value Theorem to show that such a c exists in $[0, 1]$.

   First, not that g is continuous on the interval $[0, 1]$ since the functions f and $y = x$ are continuous on that interval.

   Then note that $g(0) = f(0)$. Since $0 \le f(x) \le 1$ for all x in $[0, 1]$, it follows that $0 \le g(0) \le 1$.

   Finally, note that $g(1) = f(1) - 1$. Since $0 \le f(x) \le 1$ for all x in $[0, 1]$, it follows that $-1 \le g(1) \le 0$.

   So either $g(0) = 0$, in which case $c = 0$, or $g(1) = 0$, in which case $c = 1$, or $g(0) > 0$ and $g(1) < 0$, in which case the Intermediate Value Theorem says that there must be some number c in $[0, 1]$ for which $g(c) = 0$. Done!

3. Note that

   $$f'(0) = \lim_{x \to 0} \frac{f(x) - 0}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x \to 0} x \sin \frac{1}{x}.$$

   Since $-1 \le \sin \dfrac{1}{x} \le 1$ and $-|x| \le x \le |x|$, it follows that $-|x| \le x \sin \dfrac{1}{x} \le |x|$. Since $\lim_{x \to 0} -|x| = 0$ and $\lim_{x \to 0} |x| = 0$, it follows from the Squeeze Theorem that

   $$f'(0) = \lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

   Thus f is differentiable at $x = 0$.

4. (a) III.
   (b) I.
   (c) II.
   (d) IV.