Mathematics 1a, Section 4.1 Solutions

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2. a.

$$A = \pi r^2$$
$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt} = 2\pi r \frac{dr}{dt}$$

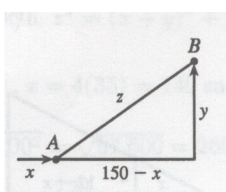
b.

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(30\text{m})(1\text{m/s}) = 60\pi\text{m}^2/\text{s}$$

6. a. Given: at noon, ship A is 150 km west of ship B; ship A is sailing east at 35 km/h, and ship B is sailing north at 25 km/h. If we let t be time (in hours), x be the distance traveled by ship A (in km), and y be the distance traveled by ship B (in km), then we are given that $dx/dt = 35$km/h and $dy/dt = 25$km/h.

b. Unknown: the rate at which the distance between the ships is changing at 4:00pm. If we let z be the distance between the ships, then we want to find dz/dt when $t = 4$h.

c.



d.

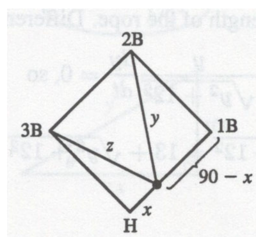
$$z^2 = (150 - x)^2 + y^2$$
$$2z\frac{dz}{dt} = 2(150 - x)\left(-\frac{dx}{dt}\right) + 2y\frac{dy}{dt}$$

e. At 4:00pm:

$$x = 4(35) = 140$$
$$y = 4(25) = 100$$
$$z = \sqrt{(150-140)^2 + 100^2} = \sqrt{10,100}$$
$$\frac{dz}{dt} = \frac{1}{z}\left[(x-150)\frac{dx}{dt} + y\frac{dy}{dt}\right] = \frac{-10(35)+100(25)}{\sqrt{10,100}} = \frac{215}{\sqrt{101}} \approx 21.4\text{km/h}$$

12. We are given that $\frac{dx}{dt} = 24$ft/s.

a.



$$y^2 = (90-x)^2 + 90^2$$
$$2y\frac{dy}{dt} = 2(90-x)\left(-\frac{dx}{dt}\right)$$

When $x = 45$, $y = \sqrt{45^2 + 90^2} = 45\sqrt{5}$, so

$$\frac{dy}{dt} = \frac{90-x}{y}\left(-\frac{dx}{dt}\right) = \frac{45}{45\sqrt{5}}(-24) = -\frac{24}{\sqrt{5}}$$

so the distance from the second base is decreasing at a rate of $\frac{24}{\sqrt{5}} \approx 10.7$ft/s.

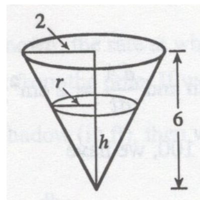
b. Due to the symmetric nature of the problem in part **a**, we expect to get the same answer - and we do.

$$z^2 = x^2 + 90^2$$
$$2z\frac{dz}{dt} = 2x\frac{dx}{dt}$$

When $x = 45$, $z = 45\sqrt{5}$, so

$$\frac{dz}{dt} = \frac{45}{45\sqrt{5}}(24) = \frac{24}{\sqrt{5}} \approx 10.7\text{ft/s}$$

18.



If C equals the rate at which water is pumped in, then

$$\frac{dV}{dt} = C - 10,000$$

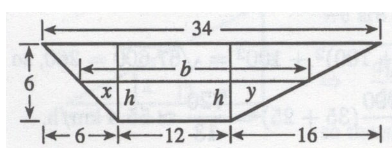
where $V = \frac{1}{3}\pi r^2 h$ is the volume at time t. By similar triangles,

$$\frac{r}{2} = \frac{h}{6}$$
$$r = \frac{1}{3}h$$
$$V = \frac{1}{3}\pi(\frac{1}{3}h)^2 h = \frac{\pi}{27}h^3$$
$$\frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$$

When $h = 200$, $\frac{dh}{dt} = 20$, so $C - 10,000 = \frac{\pi}{9}(200)^2(20)$, so

$$C = 10,000 + \frac{800,000}{9}\pi \approx 289,253\text{cm}^3/\text{min}$$

20.



$V = \frac{1}{2}(b+12)h(20) = 10(b+12)h$, and from similar triangles, $x/h = 6/6$ and $y/h = 16/6 = 8/3$, so $b = x + 12 + y = h + 12 + 8h/3 = 12 + 11h/3$. Thus

$$V = 10\left(24 + \frac{11h}{3}\right)h = 240h + \frac{110h^2}{3}$$

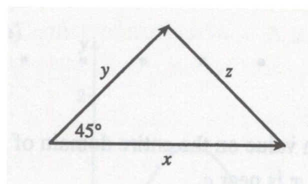
and so $0.8 = dV/dt = (240 + 220h/3)dh/dt$. When $h = 5$,

$$\frac{dh}{dt} = \frac{0.8}{240 + 5(220/3)} = \frac{3}{2275} \approx 0.00132\text{ft}/\text{min}$$

28. We want to find $\frac{dB}{dt}$ when $L = 18$ using $B = 0.007W^{2/3}$ and $W = 0.12L^{2.53}$.

$$\begin{aligned}\frac{dB}{dt} &= \frac{dB}{dW}\frac{dW}{dL}\frac{dL}{dt}\\ &= \left(0.007\cdot\frac{2}{3}W^{-1/3}\right)\left(0.12\cdot 2.53\cdot L^{1.53}\right)\left(\frac{20-15}{10,000,000}\right)\\ &= \left[0.007\cdot\frac{2}{3}(0.12\cdot 18^{2.53})^{-1/3}\right]\left(0.12\cdot 2.53\cdot 18^{1.53}\right)\left(\frac{5}{10^7}\right)\\ &\approx 1.045\times 10^{-8}\text{g/yr}\end{aligned}$$

32.



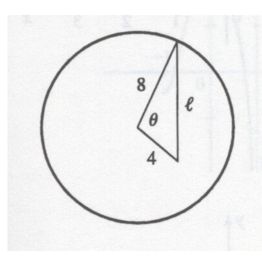
We are given that $\frac{dx}{dt} = 3$mi/h and $\frac{dy}{dt} = 2$mi/h. By the Law of Cosines,

$$\begin{aligned}z^2 &= x^2 + y^2 - 2xy\cos 45^\circ = x^2 + y^2 - \sqrt{2}xy\\ 2z\frac{dz}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - \sqrt{2}x\frac{dy}{dt} - \sqrt{2}y\frac{dx}{dt}\end{aligned}$$

After 15 minutes (which is $\frac{1}{4}$ hours), we have $x = 3/4$ and $y = 2/4 = 1/2$, so

$$\begin{aligned}z^2 &= \left(\frac{3}{4}\right)^2 + \left(\frac{1}{2}\right)^2 - \sqrt{2}\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\\ z &= \frac{1}{4}\sqrt{13-6\sqrt{2}}\\ \frac{dz}{dt} &= \frac{2}{\sqrt{13-6\sqrt{2}}}\left[2\left(\frac{3}{4}\right)3 + 2\left(\frac{1}{2}\right)2 - \sqrt{2}\left(\frac{3}{4}\right)2 - \sqrt{2}\left(\frac{1}{2}\right)3\right] = \sqrt{13-6\sqrt{2}}\end{aligned}$$

34.



The hour hand of a clock goes around once every 12 hours or, in radians per hour, $\frac{2\pi}{12} = \frac{\pi}{6}$rad/h. The minute hand goes around once an hour, or at the rate of 2πrad/h. So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of $d\theta/dt = \pi/6 - 2\pi = -11\pi/6$rad/h. Now, to relate θ to ℓ, we use the Law of Cosines: $\ell^2 = 4^2 + 8^2 - 2 \cdot 4 \cdot 8 \cdot \cos\theta = 80 - 64\cos\theta$. Differentiating implicitly with respect to t, we get

$$2\ell\frac{d\ell}{dt} = -64(-\sin\theta)\frac{d\theta}{dt}$$

At 1:00, the angle between the two hands is one-twelfth of the circle, that is, $2\pi/12 = \pi/6$ radians. We use the result of our application of the Law of Cosines to find ℓ at 1:00, and we obtain $\ell = \sqrt{80 - 64\cos\pi/6} = \sqrt{80 - 32\sqrt{3}}$. Substituting, we get

$$2\ell\frac{d\ell}{dt} = 64\sin\frac{\pi}{6}\left(-\frac{11\pi}{6}\right)$$
$$\frac{d\ell}{dt} = \frac{64(1/2)(-11\pi/6)}{2\sqrt{80 - 32\sqrt{3}}} = -\frac{88\pi}{3\sqrt{80 - 32\sqrt{3}}} \approx -18.6$$

So at 1:00, the distance between the tips of the hands is decreasing at a rate of 18.6mm/h, which is about 0.005mm/s.