Mathematics 1a, Section 4.6 Solutions

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6. If the rectangle has dimensions x and y, then its area is $xy = 1000\text{m}^2$, so $y = 1000/x$. The perimeter is

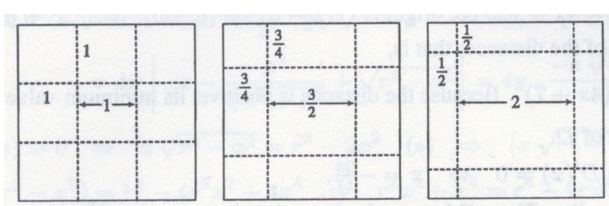
$$P = 2x + 2y = 2x + \frac{2000}{x}$$

for $x > 0$. We compute

$$P'(x) = 2 - \frac{2000}{x^2} = \frac{2}{x^2}(x^2 - 1000)$$
$$P''(x) = \frac{4000}{x^3} > 0$$

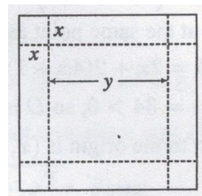
So the only critical number is $x = 1000 = 10\sqrt{10}$, and $P(\sqrt{1000}) = 40\sqrt{10}$ is a minimum, since $P''(x) > 0$. The dimensions of the rectangle with minimum perimeter are $x = y = 10\sqrt{10}\text{m}$ (it is a square).

8. a.



Three possibilities are squares of side 1, $\frac{3}{4}$, and $\frac{1}{2}$ft, which have resulting box volumes of 1, 1.6875, and 2ft^3, respectively.

b. Let x be the length of the side of the square being cut out, and let y be the length of the base. So across any edge, we fold at x and $x + y = (2x + y) - x$.



c. Volume is the product of length, width, and height. For our choice of x and y, $V = xy^2$.

d. Length of the cardboard is 3, so $x + y + x = 3$, so $y + 2x = 3$.

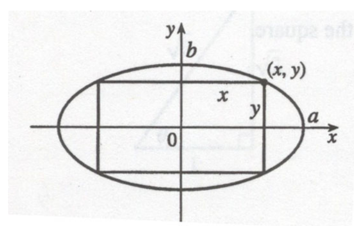
e. $y = 3 - 2x$, so $V(x) = x(3 - 2x)^2$.

f.

$$\begin{aligned} V(x) &= x(3-2x)^2 \\ V'(x) &= 2x(3-2x)(-2) + (3-2x)^2 \cdot 1 = 3(3-2x)(1-2x) \end{aligned}$$

so the critical numbers are $x = \frac{1}{2}$ and $x = \frac{3}{2}$. Now $0 \leq x \leq \frac{3}{2}$ and $V(0) = V\left(\frac{3}{2}\right) = 0$, so the maximum is $V\left(\frac{1}{2}\right) = \frac{1}{2}(2)^2 = 2\text{ft}^3$, which is the value found from the third choice of dimensions in part **a**.

18.

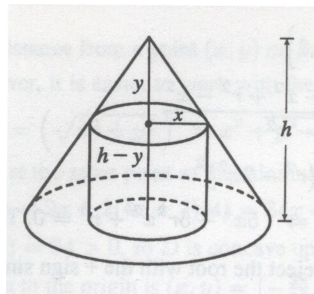


The area of the rectangle is $(2x)(2y) = 4xy$. Now $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ gives $y = \frac{b}{a}\sqrt{a^2 - x^2}$, so we maximize $A(x) = 4\frac{b}{a}x\sqrt{a^2 - x^2}$.

$$\begin{aligned} A'(x) &= \frac{4b}{a}\left[x\frac{1}{2}(a^2 - x^2)^{-1/2}(-2x) + (a^2 - x^2)^{1/2}1\right] \\ &= \frac{4b}{a}(a^2 - x^2)^{-1/2}\left[-x^2 + a^2 - x^2\right] \\ &= \frac{4b}{a\sqrt{a^2 - x^2}}[a^2 - 2x^2] \end{aligned}$$

So the critical number is $x = \frac{a}{\sqrt{2}}$, and this clearly gives a maximum. Then $y = \frac{b}{\sqrt{2}}$, so the maximum area is $4\left(\frac{a}{\sqrt{2}}\right)\left(\frac{b}{\sqrt{2}}\right) = 2ab$.

20.



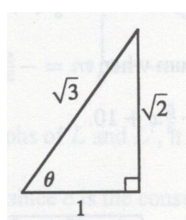
By similar triangles, $y/x = h/r$, so $y = hx/r$. The volume of the cylinder is $\pi x^2(h-y) = \pi hx^2 - (\pi h/r)x^3 = V(x)$. Now $V'(x) = 2\pi hx - (3\pi h/r)x^2 = \pi hx(2 - 3x/r)$. So $V'(x) = 0$ when $x = 0$ or $x = \frac{2}{3}r$. The maximum clearly occurs when $x = \frac{2}{3}r$, and then the volume is $\pi hx^2 - (\pi h/r)x^3 = \pi hx^2(1 - x/r) = \pi\left(\frac{2}{3}r\right)^2 h\left(1 - \frac{2}{3}\right) = \frac{4}{27}\pi r^2 h$.

25. $S = 6sh - \frac{3}{2}s^2\cot\theta + 3s^2\frac{\sqrt{3}}{2}\csc\theta$.

a. $\frac{dS}{d\theta} = \frac{3}{2}s^2\csc^2\theta - 3s^2\frac{\sqrt{3}}{2}\csc\theta\cot\theta$ or $\frac{3}{2}s^2\csc\theta(\csc\theta - \sqrt{3}\cot\theta)$.

b. $\frac{dS}{d\theta} = 0$ when $\csc\theta - \sqrt{3}\cot\theta = 0$, so $\frac{1}{\sin\theta} - \sqrt{3}\frac{\cos\theta}{\sin\theta} = 0$, so $\cos\theta = \frac{1}{\sqrt{3}}$, which occurs at $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55°$.

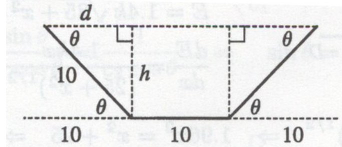
c.



If $\cos\theta = \frac{1}{\sqrt{3}}$, then $\cot\theta = \frac{1}{\sqrt{2}}$ and $\csc\theta = \frac{\sqrt{3}}{\sqrt{2}}$, so the surface area is

$$S = 6sh - \frac{3}{2}s^2\frac{1}{\sqrt{2}} + 3s^2\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{\sqrt{2}} = 6sh - \frac{3}{2\sqrt{2}}s^2 + \frac{9}{2\sqrt{2}}s^2 = 6s\left(h + \frac{s}{2\sqrt{2}}\right)$$

38.



We maximize the cross-sectional area:

$$
\begin{aligned}
A(\theta) &= 10h + 2\left(\frac{1}{2}dh\right) = 10h + dh = 10(10\sin\theta) + (10\cos\theta)(10\sin\theta) \\
&= 100(sin\theta + \sin\theta\cos\theta) \\
A'(\theta) &= 100(\cos\theta + \cos^2\theta - \sin^2\theta) = 100(\cos\theta + 2\cos^2\theta - 1) = 100(2\cos\theta - 1)(\cos\theta + 1)
\end{aligned}
$$

So we see $A'(\theta) = 0$ when $\cos\theta = \frac{1}{2}$, which occurs when $\theta = \frac{\pi}{3}$. Now $A(0) = 0$, $A\left(\frac{\pi}{2}\right) = 100$, and $A\left(\frac{\pi}{3}\right) = 75\sqrt{3} \approx 129.9$, so the maximum occurs when $\theta = \frac{\pi}{3}$.

42. a. Let D be the point such that $a = |AD|$. From the figure, $\sin\theta = \frac{b}{|BC|}$, so $|BC| = b\csc\theta$ and $\cos\theta = \frac{|BD|}{|BC|} = \frac{a-|AB|}{|BC|}$, so $|BC| = (a - |AB|)\sec\theta$. Eliminating $|BC|$ gives $(a - |AB|)\sec\theta = b\csc\theta$, so $|AB| = a - b\cot\theta$. The total resistance is

$$R(\theta) = C\frac{|AB|}{r_1^4} + C\frac{|BC|}{r_2^4} = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$

b.

$$R'(\theta) = C\left(\frac{b\csc^2\theta}{r_1^4} - \frac{b\csc\theta\cot\theta}{r_2^4}\right) = bC\csc\theta\left(\frac{\csc\theta}{r_1^4} - \frac{\cot\theta}{r_2^4}\right)$$

So we see $R'(\theta) = 0$ when $\frac{r_2^4}{r_1^4} = \frac{\cot\theta}{\csc\theta} = \cos\theta$. Also, $R'(\theta) > 0$ for $\cos\theta < \frac{r_2^4}{r_1^4}$ and $R'(\theta) < 0$ when $\cos\theta > \frac{r_2^4}{r_1^4}$, so there is an absolute minimum when $\cos\theta = \frac{r_2^4}{r_1^4}$.

c. When $r_2 \frac{2}{3} r_1$, we have $\cos\theta = \left(\frac{2}{3}\right)^4$, so $\theta = \cos^{-1}\left(\frac{2}{3}\right)^4 \approx 79°$.

44. a. $I(x) \propto \frac{\text{strength}}{(\text{distance})^2}$, and we add the intensities from the left and right light bulbs to get

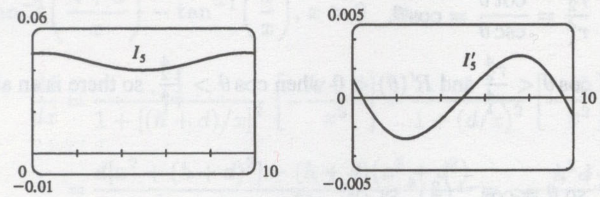
$$I(x) = \frac{k}{x^2 + d^2} + \frac{k}{(10 - x)^2} + d^2 = \frac{k}{x^2 + d^2} + \frac{k}{x^2 - 20x + 100 + d^2}$$

b. The magnitude of constant k won't affect the location of the point of maximum intensity, so for convenience we take $k = 1$.

$$I'(x) = -\frac{2x}{(x^2 + d^2)^2} - \frac{2(x - 10)}{(x^2 - 20x + 100 + d^2)^2}$$

Substituting in $d = 5$ for $I(x)$ and $I'(x)$, we get

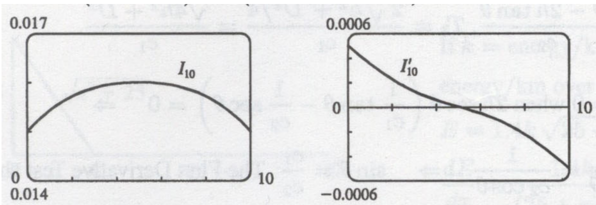
$$
\begin{aligned}
I_5(x) &= \frac{1}{x^2 + 25} + \frac{1}{x^2 - 20x + 125} \\
I_5'(x) &= -\frac{2x}{(x^2 + 25)^2} - \frac{2(x - 10)}{(x^2 - 20x + 25)^2}
\end{aligned}
$$



From the graphs, it appears that $I_5(x)$ has a minimum at $x = 5$ m.

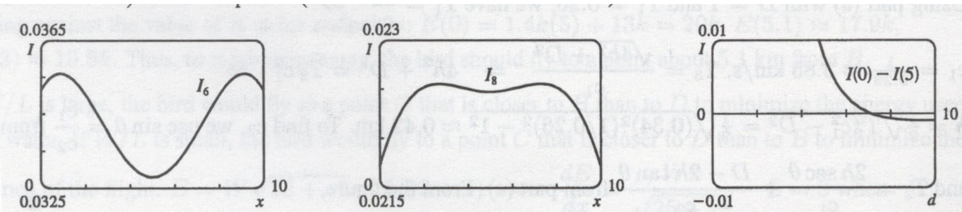
c. Substituting $d = 10$ into the equations for $I(x)$ and $I'(x)$ gives

$$I_{10}(x) = \frac{1}{x^2 + 100} + \frac{1}{x^2 - 20x + 200}$$
$$I'_{10}(x) = -\frac{2x}{(x^2 + 100)^2} - \frac{2(x - 10)}{(x^2 - 20x + 200)^2}$$



From the graphs, it seems that for $d = 10$, the intensity is minimized at the endpoints, that is, $x = 0$ and $x = 10$. The midpoint is now the most brightly lit point!

d. From the graphs of the above functions, we see that the minimal illumination changes from the midpoint ($x = 5$ with $d = 5$) to the endpoints ($x = 0$ and $x = 10$ with $d = 10$).



So we try $d = 6$ and from the graph see the minimum value still occurs at $x = 5$. Next, we let $d = 8$ and the minimum value occurs at the endpoints. It appears that for some value of d between 6 and 8, we must have minima at both the midpoint and the endpoints, that is, $I(5)$ must equal $I(0)$. To find this value of d, we solve $I(0) = I(5)$, with $k = 1$.

$$\frac{1}{d^2} + \frac{100 + d^2}{=} \frac{1}{25 + d^2} + \frac{1}{25 + d^2} = \frac{2}{25 + d^2}$$
$$(25 + d^2)(100 + d^2) + d^2(25 + d^2) = 2d^2(100 + d^2)$$
$$2500 + 125d^2 + d^4 + 25d^2 + d^4 = 200d^2 + 2d^4$$
$$2500 = 50d^2$$
$$d = 5\sqrt{2} \approx 7.071$$

Thus the point of minimal illumination changes abruptly from the midpoint to the endpoints when $d = 5\sqrt{2}$.