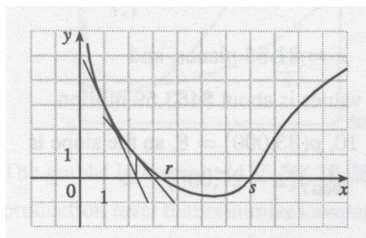
Mathematics 1a, Section 4.8 Solutions

Alexander Ellis

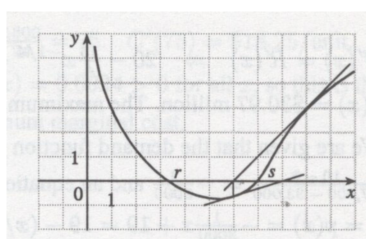
December 6, 2004

1.



The tangent line at $x = 1$ intersects the x-axis at $x \approx 2.3$, so $x_2 \approx 2.3$. The tangent line at $x = 2.3$ intersects the x-axis at $x \approx 3$, so $x_3 \approx 3.0$.

2.



The tangent line at $x = 9$ intersects the x-axis at $x \approx 6.0$, so $x_2 \approx 6.0$. The tangent line at $x = 6.0$ intersects the x-axis at $x \approx 8.0$, so $x_3 \approx 8.0$.

6. $f(x) = x^3 - x^2 - 1$, so $f'(x) = 3x^2 - 2x$, so

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - x_n^2 - 1}{3x_n^2 - 2x_n}$$

Now $x_1 = 1$, so $x_2 = 1 - \frac{1-1-1}{3-2} = 2$, and $x_3 = 2 - \frac{2^3-2^2-1}{3\cdot 2^2 - 2\cdot 2} = 1.625$.

10. $f(x) = x^4 + x - 4$, so $f'(x) = 4x^3 + 1$. Thus $x_{n+1} = x_n - \frac{x_n^4 + x_n - 4}{4x_n^3 + 1}$. $x_1 = 1.5$, so $x_2 \approx 1.323276$, $x_3 \approx 1.285346$, $x_4 \approx 1.283784$, $x_5 \approx 1.283782 \approx x_6$. So the root is 1.283782, to six decimal places.

22.

$$
\begin{aligned}
x^3 - x &= 1 \\
x^3 - x - 1 &= - \\
f(x) &= x^3 - x - 1 \\
f'(x) &= 3x^2 - 1 \\
x_{n+1} &= x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}
\end{aligned}
$$

a.

$$
\begin{aligned}
x_1 &= 1 \\
x_2 &= 1.5 \\
x_3 &\approx 1.347826 \\
x_4 &\approx 1.325200 \\
x_5 &\approx 1.324718 \approx x_6
\end{aligned}
$$

b.

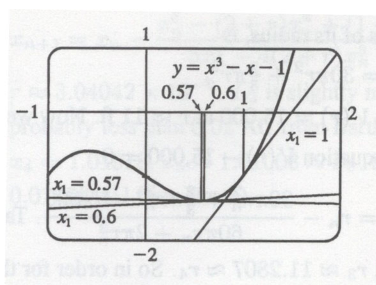
$$
\begin{aligned}
x_1 &= 0.6 \\
x_2 &= 17.9 \\
x_3 &\approx 11.946802 \\
x_4 &\approx 7.985520 \\
x_5 &\approx 5.356909 \\
x_6 &\approx 3.624996 \\
x_7 &\approx 2.505589 \\
x_8 &\approx 1.820129 \\
x_9 &\approx 1.461044 \\
x_{10} &\approx 1.339323 \\
x_{11} &\approx 1.324913 \\
x_{12} &\approx 1.324718 \approx x_{13}
\end{aligned}
$$

c.

$$x_1 = 0.57$$
$$x_2 \approx -54.165455$$
$$x_3 \approx -36.114293$$
$$x_4 \approx -24.082094$$
$$x_5 \approx -16.063387$$
$$x_6 \approx -10.721483$$
$$x_7 \approx -7.165534$$
$$x_8 \approx -4.801704$$
$$x_9 \approx -3.233425$$
$$x_{10} \approx -2.193674$$
$$x_{11} \approx -1.496867$$
$$x_{12} \approx -0.997546$$
$$x_{13} \approx -0.496305$$
$$x_{14} \approx -2.894162$$
$$x_{15} \approx -1.967962$$
$$x_{16} \approx -1.341355$$
$$x_{17} \approx -0.870187$$
$$x_{18} \approx -0.249949$$
$$x_{19} \approx -1.192219$$
$$x_{20} \approx -0.731952$$
$$x_{21} \approx 0.355213$$
$$x_{22} \approx -1.753322$$
$$x_{23} \approx -1.189420$$
$$x_{24} \approx -0.729123$$
$$x_{25} \approx 0.377844$$
$$x_{26} \approx -1.937872$$
$$x_{27} \approx -1.320350$$

$$x_{28} \approx -0.851919$$
$$x_{29} \approx -0.200959$$
$$x_{30} \approx -1.119386$$
$$x_{31} \approx -0.654291$$
$$x_{32} \approx 1.547010$$
$$x_{33} \approx 1.360051$$
$$x_{34} \approx 1.325828$$
$$x_{35} \approx 1.324719$$
$$x_{36} \approx 1.324718 \approx x_{37}$$

d.

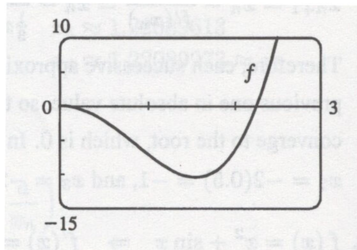


From the figure, we see that the tangent line corresponding to $x_1 = 1$ results in a sequence of approximations that converges quite quickly ($x_5 \approx x_6$). The tangent line corresponding to $x_1 = 0.6$ is close to being horizontal, so x_2 is quite far away from the root. But the sequence still converges - just a little more slowly ($x_{12} \approx x_{13}$). Lastly, the tangent line corresponding to $x_1 = 0.57$ is very nearly horizontal, x_2 is farther away from the root, and the sequence takes more iterations to converge ($x_{36} \approx x_{37}$).

28. Let the radius of the circle be r. Using $s = r\theta$, we have $5 = r\theta$ and so $r = 5/\theta$. From the Law of Cosines we get $4^2 = r^2 + r^2 - 2r^2 \cos\theta$, so $16 = 2r^2(1-\cos\theta) = 2(5/\theta)^2(1-\cos\theta)$. Multiplying by θ^2 gives $16\theta^2 = 50(1-\cos\theta)$, so we take $f(\theta) = 16\theta^2 + 50\cos\theta - 50$ and so $f'(\theta) = 32\theta - 50\sin\theta$. Newton's method gives us

$$\theta_{n+1} = \theta_n - \frac{16\theta_n^2 + 50\cos\theta_n - 50}{32\theta_n - 50\sin\theta_n}$$

From the graph of f, we can use $\theta_1 = 2.2$, giving us $\theta_2 \approx 2.2662$ and $\theta_3 \approx 2.622 \approx \theta_4$. So correct to four decimal places, the angle is 2.622 radians $\approx 130°$.



30. a.

$$
\begin{aligned}
p(x) &= x^5 - (2+r)x^4 + (1+2r)x^3 - (1-r)x^2 + 2(1-r)x + r - 1 \\
p'(x) &= 5x^4 - 4(2+r)x^3 + 3(1+2r)x^2 - 2(1-r)x + 2(1-r) \\
x_{n+1} &= x_n - \frac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1-r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1-r)x_n + 2(1-r)}
\end{aligned}
$$

We substitute the value $r \approx 3.04042 \times 10^{-6}$ in order to evaluate the approximations numerically. The libration point L_1 is slightly less than 1 AU from the sun, so we take $x_1 = 0.95$ as our approximation, and get

$$
\begin{aligned}
x_2 &\approx 0.96682 \\
x_3 &\approx 0.97770 \\
x_4 &\approx 0.98451 \\
x_5 &\approx 0.98830 \\
x_6 &\approx 0.98976 \\
x_7 &\approx 0.98998 \\
x_8 &\approx 0.98999 \approx x_9
\end{aligned}
$$

So, to five decimal places, L_1 is located 0.98999 AU from the sun (or 0.01001 AU from Earth).

b. In this case we use Newton's method with the function

$$
\begin{aligned}
p(x) - 2rx^2 &= x^5 - (2+r)x^4 + (1+2r)x^3 - (1+r)x^2 + 2(1-r)x + r - 1 \\
\left[p(x) - 2rx^2\right]' &= 5x^4 - 4(2+r)x^3 + 3(1+2r)x^2 - 2(1+r)x + 2(1-r) \\
x_{n+1} &= x_n - \frac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1+r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1+r)x_n + 2(1-r}
\end{aligned}
$$

Again, we substitute $r \approx 3.04042 \times 10^{-6}$. L_2 is slightly more than 1 AU from the sun and, judging from the result of part **a**, probably less than 0.02 AU from Earth. So we take

$x_1 = 1.02$ and get

$$\begin{aligned} x_2 &\approx 1.01422 \\ x_3 &\approx 1.01118 \\ x_4 &\approx 1.01018 \\ x_5 &\approx 1.01008 \approx x_6 \end{aligned}$$

So, to five decimal places, L_2 is located 1.01008 AU from the sun (or 0.01008 AU from Earth).