Mathematics 1a, Section 5.4 Solutions

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2. a. $g(x) = \int_0^x f(t)dt$, so $g(0) = \int_0^0 f(t)dt = 0$. To continue, we break up the integrals as $g(n) = \int_0^n f(t)dt = \int_0^{n-1} f(t)dt + \int_{n-1}^n f(t)dt = g(n-1) + \int_{n-1}^n f(t)dt$, where the second integral can be evaluated as the area of a triangle:

$$g(1) = \int_0^1 f(t)dt = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

$$g(2) = g(1) + \int_1^2 f(t)dt = \frac{1}{2} - \frac{1}{2} \cdot 1 \cdot 1 = 0$$

$$g(3) = g(2) + \int_2^3 f(t)dt = 0 - \frac{1}{2} \cdot 1 \cdot 1 = -\frac{1}{2}$$

$$g(4) = g(3) + \int_3^4 f(t)dt = -\frac{1}{2} + \frac{1}{2} \cdot 1 \cdot 1 = 0$$

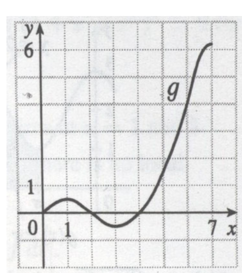
$$g(5) = g(4) + \int_4^5 f(t)dt = 0 + 1.5 = 1.5$$

$$g(6) = g(5) + \int_5^6 f(t)dt = 1.5 + 2.5 = 4$$

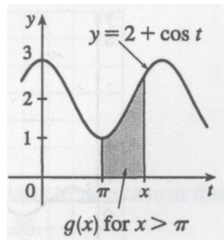
b. $g(7) = g(6) + \int_6^7 f(t)dt \approx 4 + 2.2 = 6.2$ (we have estimated from the graph).

c. The answers from part **a** and part **b** indicate that g has a minimum at $x = 3$ and a maximum at $x = 7$. This makes sense from the graph of f since we are subtracting area on $1 < x < 3$ and adding area on $3 < x < 7$.

d.



6.



a. By FTC1, $g(x) = \int_{\pi}^{x}(2+\cos t)dt$, so $g'(x) = f(x) = 2 + \cos x$.

b. By FTC2, $g(x) = \int_{\pi}^{x}(2+\cos t)dt = [2t + \sin t]_{\pi}^{x} = (2x + \sin x) - (2\pi + 0) = 2x + \sin x - 2\pi$, so $g'(x) = 2 + \cos x$.

14. Let $u = e^x$. Then $\frac{du}{dx} = e^x$. Also, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, so

$$\begin{aligned} y' &= \frac{d}{dx}\int_{e^x}^{0} \sin^3 t dt \\ &= \frac{d}{du}\int_{u}^{0} \sin^3 t dt \frac{du}{dx} = -\frac{d}{du}\int_{0}^{u} \sin^3 t dt \frac{du}{dx} \\ &= -\sin^3 u \cdot e^x = -e^x \sin^3(e^x) \end{aligned}$$

18. For the curve to be concave upward, we must have $y'' > 0$.

$$\begin{aligned} y &= \int_0^x \frac{1}{1+t+t^2} dt \\ y' &= \frac{1}{1+x+x^2} \\ y'' &= \frac{-(1+2x)}{(1+x+x^2)^2} \end{aligned}$$

For this expression to be positive, we must have $(1+2x) < 0$, since $(1+x+x^2) > 0$ for all x. $(1+2x) < 0$ when $x < -\frac{1}{2}$. Thus, the curve is concave upward on $(-\infty, -1/2)$.

19. a. By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 1, 3, 5, 7, 9$. g has local maxima at $x = 1$ and 5 (since $f = g'$ changes from positive to negative there) and local minima at $x = 3$ and 7. There is no local maximum or minimum at $x = 9$, since f is not defined for $x > 9$.

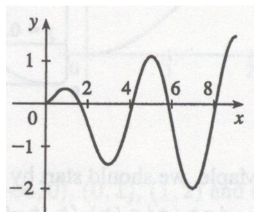
b. We can see from the graph that

$$|\int_0^1 f dt| < |\int_1^3 f dt| < |\int_3^5 f dt| < |\int_5^7 f dt| < |\int_7^9 f dt|$$

So $g(1) = |\int_0^1 f dt|$, $g(5) = \int_0^5 f dt = g(1) - |\int_1^3 f dt| + |\int_3^5 f dt|$, and $g(9) = \int_0^9 f dt = g(5) - |\int_5^7 f dt| + |\int_7^9 f dt|$. Thus, $g(1) < g(5) < g(9)$, and so the absolute maximum of $g(x)$ occurs at $x = 9$.

c. g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on (approximately) $(1/2, 2)$, $(4, 6)$, and $(8, 9)$. So g is concave downward on these intervals.

d.



20. a. By the first fundamental theorem of calculus, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 2, 4, 6, 8, 10$. g has a local maxima at $x = 2, 6$, since that's where $f = g'$ changes from positive to negative. g has local minima at $x = 4, 8$. There is no local maximum or minimum at $x = 10$, since f is not defined for $x > 10$.

b. We can see from the graph that

$$|\int_0^2 f dt| > |\int_2^4 f dt| > |\int_4^6 f dt| > |\int_6^8 f dt| > |\int_8^1 0 f dt|$$

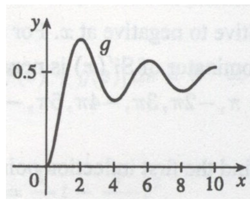
So then

$$\begin{aligned} g(2) &= |\int_0^2 f dt| \\ g(6) &= \int_0^6 f dt = g(2) - |\int_2^4 f dt| + |\int_4^6 f dt| \\ g(10) &= \int_0^{10} f dt = g(6) - |\int_6^8 f dt| + |\int_8^{10} f dt| \end{aligned}$$

Thus, $g(2) > g(6) > g(10)$, and so the absolute maximum of $g(x)$ occurs at $x = 2$.

c. g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on $(1, 3)$, $(5, 7)$, and $(9, 10)$. So g is concave downard on these intervals.

d.



24. a. If $x < 0$, then

$$g(x) = \int_0^x f(t)dt = \int_0^x 0dt = 0$$

If $0 \leq x \leq 1$, then

$$g(x) = \int_0^x f(t)dt = \int_0^x tdt = \left[\frac{1}{2}t^2\right]_0^x = \frac{1}{2}x^2$$

If $1 \leq x \leq 2$, then

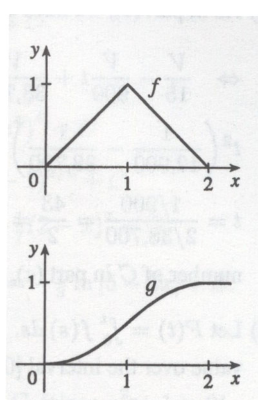
$$\begin{aligned} g(x) = & \int_0^x f(t)dt = \int_0^1 f(t)dt + \int_1^x f(t)dt \\ = & \; g(1) + \int_1^x (2-t)dt = \frac{1}{2}(1)^2 + \left[2t - \frac{1}{2}t^2\right]_1^x \\ = & \; \frac{1}{2} + \left(2x - \frac{1}{2}x^2\right) - \left(2 - \frac{1}{2}\right) = 2x - \frac{1}{2}x^2 - 1 \end{aligned}$$

If $x > 2$, then

$$g(x) = \int_0^x f(t)dt = g(2) + \int_2^x 0dt = 1 + 0 = 1$$

So $g(x) = 0$ for $x < 0$, $\frac{1}{2}x^2$ for $0 \leq x \leq 1$, $2x - \frac{1}{2}x^2 - 1$ for $1 < x \leq 2$, and 1 for $x > 2$.

b.



c. f is not differentiable at its corners at $x = 0, 1, 2$. f is differentiable on $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, and $(2, \infty)$. g is differentiable on $(-\infty, \infty)$.

26. a. $C(t) = \frac{1}{t}\int_0^t [f(s) + g(s)]ds$. Using the first fundamental theorem of calculus and the product rule, we have $C'(t) = \frac{1}{t}[f(t) + g(t)] - \frac{1}{t^2}\int_0^t [f(s) + g(s)]ds$. Set $C'(t) = 0$. Then set equal to zero:

$$\frac{1}{t}[f(t) + g(t)] - \frac{1}{t^2}\int_0^t [f(s) + g(s)]ds = 0$$
$$[f(t) + g(t)] - \frac{1}{t}\int_0^t [f(s) + g(s)]ds = 0$$
$$[f(t) - g(t)] - C(t) = 0$$
$$C(t) = f(t) + g(t)$$

b. For $0 \le t \le 30$, we have

$$D(t) = \int_0^t \left(\frac{V}{15} - \frac{V}{450}s\right) ds - \left[\frac{V}{15}s - \frac{V}{900}s^2\right]_0^t = \frac{V}{15}t - \frac{V}{900}t^2$$

Set $D(t) = V$ and we get:

$$\frac{V}{15}t - \frac{V}{900}t^2 = V$$
$$60t - t^2 = 900$$
$$t^2 - 60t + 900 = 0$$
$$(t - 30)^2 = 0$$
$$t = 30$$

So the length of time T is 30 months.

c.

$$C(t) = \frac{1}{t}\int_0^t \left(\frac{V}{15} - \frac{V}{450}s + \frac{V}{12,900}s^2\right) ds = \frac{1}{t}\left[\frac{V}{15}s - \frac{V}{900}s^2 + \frac{V}{38,700}s^3\right]_0^t$$
$$= \frac{1}{t}\left(\frac{V}{15}t - \frac{V}{900}t^2 + \frac{V}{38,700}t^3\right) = \frac{V}{15} - \frac{V}{900}t + \frac{V}{38,700}t^2$$
$$C'(t) = -\frac{V}{900} + \frac{V}{19,350}t$$

We see that $C'(t) = 0$ when $\frac{1}{19,350}t = \frac{1}{900}$, that it, $t = 21.5$. So find the minimum, we look here and at the endpoints:

$$
\begin{aligned}
C(21.5) &= \frac{V}{15} - \frac{V}{900}(21.5) + \frac{V}{38,700}(21.5)^2 \approx 0.05472V \\
C(0) &= \frac{V}{15} \approx 0.06667V \\
C(30) &= \frac{V}{15} - \frac{V}{900}(30) + \frac{V}{38,700}(30)^2 \approx 0.05659V
\end{aligned}
$$

So the absolute minimum $C(21.5) \approx 0.05472V$.

d. As in part **c**, we have $C(t) = \frac{V}{15} - \frac{V}{900}t + \frac{V}{38,700}t^2$. Now set $C(t) = f(t) + g(t)$.

$$
\begin{aligned}
\frac{V}{15} - \frac{V}{900}t + \frac{V}{38,700}t^2 &= \frac{V}{15} - \frac{V}{450}t + \frac{V}{12,900}t^2 \\
t^2\left(\frac{1}{12,900} - \frac{1}{38,700}\right) &= t\left(\frac{1}{450} - \frac{1}{900}\right) \\
t = \frac{1/900}{2/38,700} &= \frac{43}{2} = 21.5
\end{aligned}
$$

This is the value of t that we obtained as the critical number of C in part **c**, so we have verified the result of **a** in this case.