Name: __________________ Class:___ Date: _____________

1. $f(x) = 10 + 27x - x^3$, $0 \le x \le 4$. $f'(x) = 27 - 3x^2 = -3(x^2 - 9) = -3(x+3)(x-3) = 0$ only when $x = 3$ (since -3 is not in the domain). $f'(x) > 0$ for $x < 3$ and $f'(x) < 0$ for $x > 3$, so $f(3) = 64$ is a local maximum value. Checking the endpoints, we find $f(0) = 10$ and $f(4) = 54$. Thus, $f(0) = 10$ is the absolute minimum value and $f(3) = 64$ is the absolute maximum value.

2. $f(x) = x - \sqrt{x}$, $0 \le x \le 4$. $f'(x) = 1 - 1/(2\sqrt{x}) = 0 \quad \Leftrightarrow \quad 2\sqrt{x} = 1 \quad \Rightarrow \quad x = \frac{1}{4}$. $f'(x)$ does not exist $\Leftrightarrow$ $x = 0$. $f'(x) < 0$ for $0 < x < \frac{1}{4}$ and $f'(x) > 0$ for $\frac{1}{4} < x < 4$, so $f\left(\frac{1}{4}\right) = -\frac{1}{4}$ is a local and absolute minimum value. $f(0) = 0$ and $f(4) = 2$, so $f(4) = 2$ is the absolute maximum value.

3. $f(x) = \dfrac{x}{x^2 + x + 1}$, $-2 \le x \le 0$. $f'(x) = \dfrac{(x^2 + x + 1)(1) - x(2x+1)}{(x^2 + x + 1)^2} = \dfrac{1 - x^2}{(x^2 + x + 1)^2} = 0 \quad \Leftrightarrow \quad x = -1$ (since 1 is not in the domain). $f'(x) < 0$ for $-2 < x < -1$ and $f'(x) > 0$ for $-1 < x < 0$, so $f(-1) = -1$ is a local and absolute minimum value. $f(-2) = -\frac{2}{3}$ and $f(0) = 0$, so $f(0) = 0$ is an absolute maximum value.

4. $f(x) = \dfrac{\ln x}{x^2}$, $[1, 3]$. $f'(x) = \dfrac{x^2 \cdot \frac{1}{x} - (\ln x)(2x)}{(x^2)^2} = \dfrac{x - 2x \ln x}{x^4} = \dfrac{1 - 2\ln x}{x^3} = 0 \quad \Leftrightarrow \quad \ln x = \frac{1}{2} \quad \Leftrightarrow$ $x = e^{1/2} = \sqrt{e} \approx 1.65$. $f'(x) > 0$ for $x < \sqrt{e}$ and $f'(x) < 0$ for $x > \sqrt{e}$, so f is increasing on $(1, \sqrt{e}\,)$ and decreasing on $(\sqrt{e}, 3)$. Hence, $f(\sqrt{e}\,) = \dfrac{1}{2e}$ is a local maximum value. $f(1) = 0$ and $f(3) = \dfrac{\ln 3}{9} \approx 0.12$. Since $\dfrac{1}{2e} \approx 0.18$, $f(\sqrt{e}\,) = \dfrac{1}{2e}$ is the absolute maximum value and $f(1) = 0$ is the absolute minimum value.

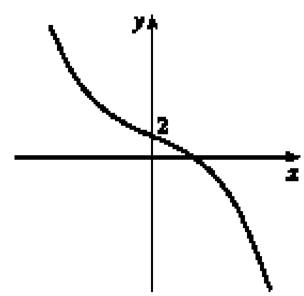
5. (a) $f(x) = 2 - 2x - x^3$ is a polynomial, so there is no asymptote.

(b) $f'(x) = -2 - 3x^2 = -1(3x^2 + 2) < 0$, so f is decreasing on $\mathbb{R}$.

(c) No local extrema

(d) $f''(x) = -6x < 0$ on $(0, \infty)$ and $f''(x) > 0$ on $(-\infty, 0)$, so f is CD on $(0, \infty)$ and CU on $(-\infty, 0)$. There is an IP at $(0, 2)$.

(e)



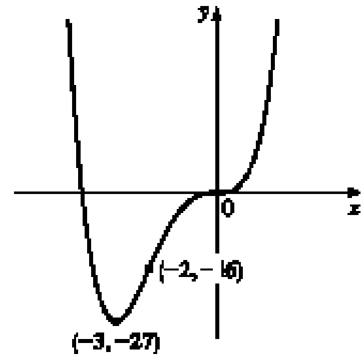
6. (a) $f(x) = x^4 + 4x^3$ is a polynomial, so there is no asymptote.

(b) $f'(x) = 4x^3 + 12x^2 = 4x^2(x+3) > 0 \quad \Leftrightarrow \quad x > -3$, so f is increasing on $(-3, \infty)$ and decreasing on $(-\infty, -3)$.

(c) Local minimum $f(-3) = -27$, no local maximum

(d) $f''(x) = 12x^2 + 24x = 12x(x+2) < 0 \quad \Leftrightarrow \quad -2 < x < 0$, so f is CD on $(-2, 0)$ and CU on $(-\infty, -2)$ and $(0, \infty)$. IP at $(0, 0)$ and $(-2, -16)$.

(e)



7. (a) $f(x) = x + \sqrt{1-x}$ has no asymptote.

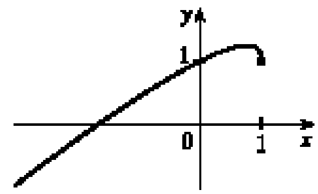
(b) $f'(x) = 1 - 1/\left(2\sqrt{1-x}\right) = 0 \quad \Leftrightarrow \quad 2\sqrt{1-x} = 1 \quad \Leftrightarrow$

$1 - x = \frac{1}{4} \quad \Leftrightarrow \quad x = \frac{3}{4}$ and $f'(x) > 0 \quad \Leftrightarrow \quad x < \frac{3}{4}$, so f is increasing on $\left(-\infty, \frac{3}{4}\right)$ and decreasing on $\left(\frac{3}{4}, 1\right)$.

(c) $f\left(\frac{3}{4}\right) = \frac{3}{4} + \sqrt{1 - \frac{3}{4}} = \frac{3}{4} + \sqrt{\frac{1}{4}} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$ is a local maximum.

(d) $f''(x) = -\dfrac{1}{4(1-x)^{3/2}} < 0$ on the domain of f, so f is CD on $(-\infty, 1)$. No IP

(e)



8. (a) $f(x) = \dfrac{1}{1-x^2} = \dfrac{1}{(1+x)(1-x)}$ has vertical asymptotes $x = \pm 1$. $\lim\limits_{x \to \pm\infty} f(x) = 0$, so f has a horizontal asymptote of $y = 0$.

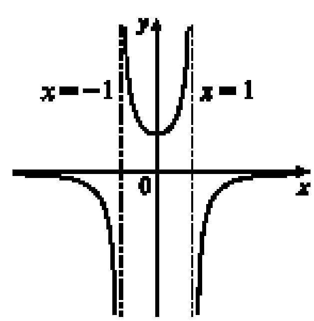
(b) $f'(x) = \dfrac{2x}{(1-x^2)^2} = 0 \quad \Leftrightarrow \quad x = 0$, so f is decreasing on $(-\infty, -1)$ and $(-1, 0)$, and increasing on $(0, 1)$ and $(1, \infty)$.

(c) Local minimum $f(0) = 1$; no local maximum

(d) $f''(x) = \dfrac{(1-x^2)^2 \cdot 2 - 2x \cdot 2(1-x^2)(-2x)}{(1-x^2)^4} = \dfrac{2(1-x^2)^2 + 8x^2}{(1-x^2)^3} =$

$\dfrac{6x^2 + 2}{(1-x^2)^3} < 0 \quad \Rightarrow \quad x^2 > 1$, so f is CD on $(-\infty, -1)$ and $(1, \infty)$, and CU on $(-1, 1)$. No IP

(e)



9. (a) $y = f(x) = \sin^2 x - 2\cos x$ has no asymptote.

(b) $y' = 2\sin x\,\cos x + 2\sin x = 2\sin x\,(\cos x + 1)$. $y' = 0 \quad\Leftrightarrow\quad \sin x = 0$ or $\cos x = -1 \quad\Leftrightarrow\quad x = n\pi$ or $x = (2n+1)\pi$. $y' > 0$ when $\sin x > 0$, since $\cos x + 1 \geq 0$ for all x. Therefore, $y' > 0$ (and so f is increasing) on $(2n\pi, (2n+1)\pi)$; $y' < 0$ (and so f is decreasing) on $((2n-1)\pi, 2n\pi)$ or equivalently, $((2n+1)\pi, (2n+2)\pi)$.

(c) Local maxima are $f((2n+1)\pi) = 2$; local minima are $f(2n\pi) = -2$.

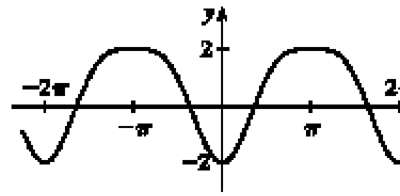
(d) $y' = \sin 2x + 2\sin x \quad\Rightarrow$

$$y'' = 2\cos 2x + 2\cos x = 2\left(2\cos^2 x - 1\right) + 2\cos x = 4\cos^2 x + 2\cos x - 2$$

$$= 2\left(2\cos^2 x + \cos x - 1\right) = 2(2\cos x - 1)(\cos x + 1)$$

$y'' = 0 \quad\Leftrightarrow\quad \cos x = \frac{1}{2}$ or $-1 \quad\Leftrightarrow\quad x = 2n\pi \pm \frac{\pi}{3}$ or $x = (2n+1)\pi$. $y'' > 0$ (and so f is CU) on $\left(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right)$; $y'' \leq 0$ (and so f is CD) on $\left(2n\pi + \frac{\pi}{3}, 2n\pi + \frac{5\pi}{3}\right)$. There are inflection points at $\left(2n\pi \pm \frac{\pi}{3}, -\frac{1}{4}\right)$.

(e)



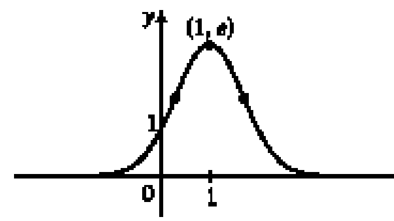
10. (a) $\lim\limits_{x\to\pm\infty} e^{2x-x^2} = 0$, so $y = 0$ is a HA.

(b) $y = f(x) = e^{2x-x^2} \quad\Rightarrow\quad f'(x) = 2(1-x)e^{2x-x^2} > 0 \quad\Leftrightarrow\quad x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.

(c) $f(1) = e$ is a local and absolute maximum.

(d) $f''(x) = 2\left(2x^2 - 4x + 1\right)e^{2x-x^2} = 0 \quad\Leftrightarrow\quad x = 1 \pm \frac{\sqrt{2}}{2}$. $f''(x) > 0$ $\Leftrightarrow\quad x < 1 - \frac{\sqrt{2}}{2}$ or $x > 1 + \frac{\sqrt{2}}{2}$, so f is CU on $\left(-\infty, 1 - \frac{\sqrt{2}}{2}\right)$ and $\left(1 + \frac{\sqrt{2}}{2}, \infty\right)$, and CD on $\left(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}\right)$. IP at $\left(1 \pm \frac{\sqrt{2}}{2}, \sqrt{e}\right)$.

(e)



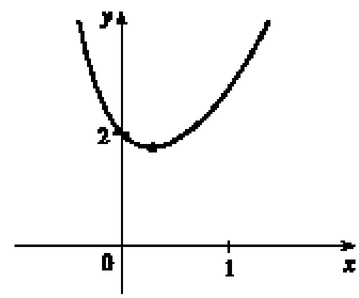
11. (a) $\lim\limits_{x\to\pm\infty} \left(e^x + e^{-3x}\right) = \infty$, no asymptote.

(b) $y = f(x) = e^x + e^{-3x} \quad\Rightarrow\quad f'(x) = e^x - 3e^{-3x} = e^{-3x}\left(e^{4x} - 3\right) > 0 \quad\Leftrightarrow$ $e^{4x} > 3 \quad\Leftrightarrow\quad 4x > \ln 3 \quad\Leftrightarrow\quad x > \frac{1}{4}\ln 3$, so f is increasing on $\left(\frac{1}{4}\ln 3, \infty\right)$ and decreasing on $\left(-\infty, \frac{1}{4}\ln 3\right)$.

(c) $f\left(\frac{1}{4}\ln 3\right) = 3^{1/4} + 3^{-3/4} \approx 1.75$ is a local and absolute minimum.

(d) $f''(x) = e^x + 9e^{-3x} > 0$, so f is CU on $(-\infty, \infty)$. No IP

(e)



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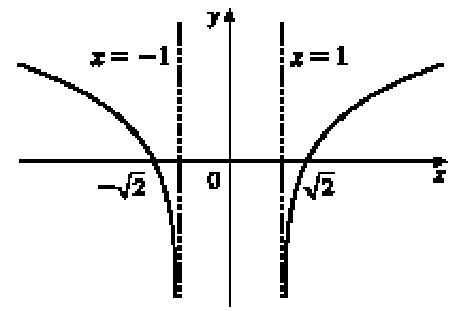
12. (a) $\lim_{x\to\pm\infty} \ln(x^2-1) = \infty$, $\lim_{x\to 1^+} \ln(x^2-1) = -\infty$, $\lim_{x\to -1^-} \ln(x^2-1) = -\infty$, so $x = 1$ and $x = -1$ are VA.

(b) $y = f(x) = \ln(x^2-1) \quad \Rightarrow \quad f'(x) = \dfrac{2x}{x^2-1} > 0$ for $x > 1$ and $f'(x) < 0$ for $x < -1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, -1)$. Note that the domain of f is $|x| > 1$.

(c) No maximum or minimum.

(d) $f''(x) = -2\dfrac{x^2+1}{(x^2-1)^2} < 0$, so f is CD on $(-\infty, -1)$ and $(1, \infty)$. No IP

(e)



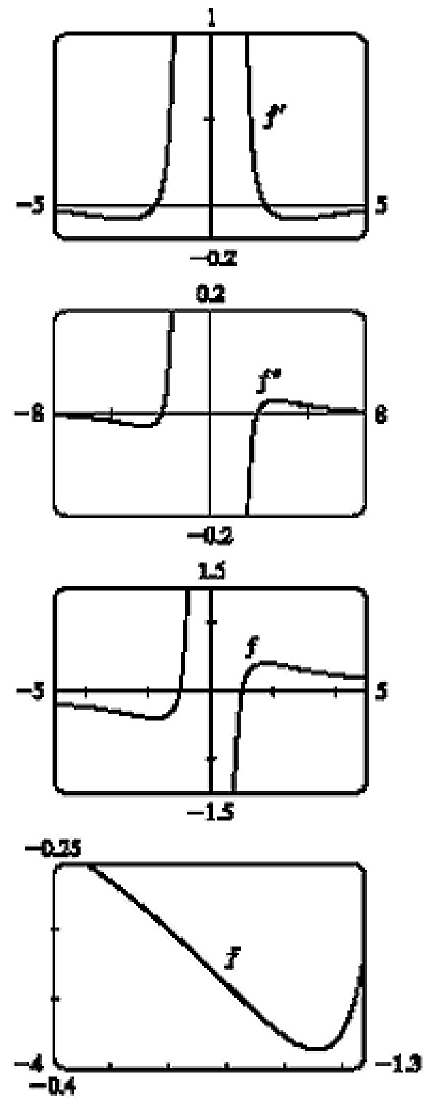
13. $f(x) = \dfrac{x^2-1}{x^3} \quad \Rightarrow \quad f'(x) = \dfrac{x^3(2x) - (x^2-1)3x^2}{x^6} = \dfrac{3-x^2}{x^4} \quad \Rightarrow$

$f''(x) = \dfrac{x^4(-2x) - (3-x^2)4x^3}{x^8} = \dfrac{2x^2-12}{x^5}$

Estimates: From the graphs of f' and f'', it appears that f is increasing on $(-1.73, 0)$ and $(0, 1.73)$ and decreasing on $(-\infty, -1.73)$ and $(1.73, \infty)$; f has a local maximum of about $f(1.73) = 0.38$ and a local minimum of about $f(-1.7) = -0.38$; f is CU on $(-2.45, 0)$ and $(2.45, \infty)$, and CD on $(-\infty, -2.45)$ and $(0, 2.45)$; and f has inflection points at about $(-2.45, -0.34)$ and $(2.45, 0.34)$.

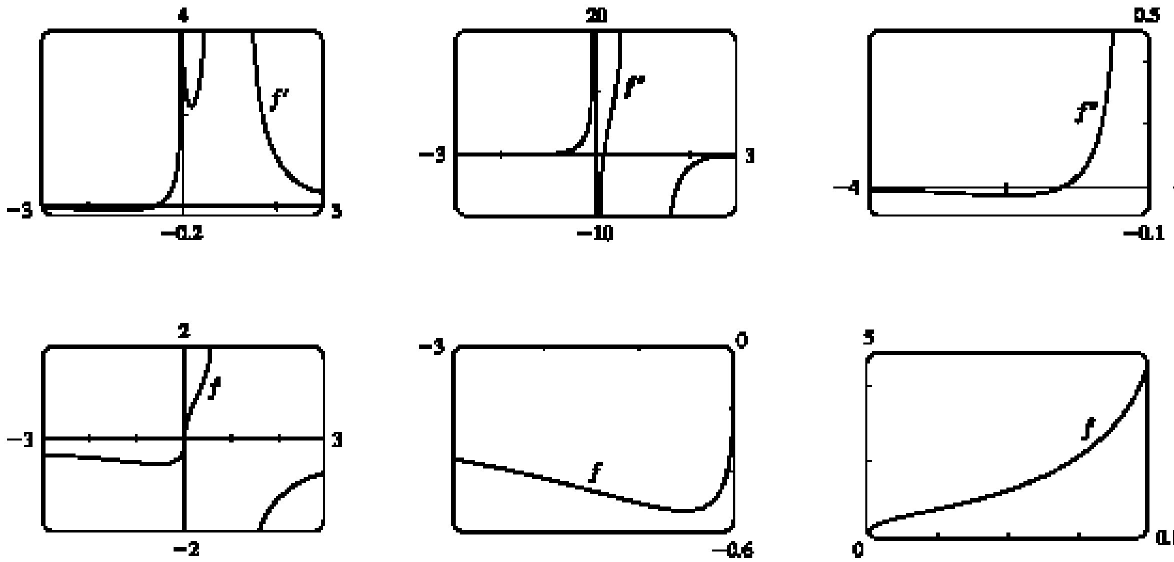
Exact: Now $f'(x) = \dfrac{3-x^2}{x^4}$ is positive for $0 < x^2 < 3$, that is, f is increasing on $(-\sqrt{3}, 0)$ and $(0, \sqrt{3})$; and $f'(x)$ is negative (and so f is decreasing) on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$. $f'(x) = 0$ when $x = \pm\sqrt{3}$.

f' goes from positive to negative at $x = \sqrt{3}$, so f has a local maximum of $f(\sqrt{3}) = \frac{(\sqrt{3})^2 - 1}{(\sqrt{3})^3} = \frac{2\sqrt{3}}{9}$; and since f is odd, we know that maxima on the interval $(0, \infty)$ correspond to minima on $(-\infty, 0)$, so f has a local minimum of $f(-\sqrt{3}) = -\frac{2\sqrt{3}}{9}$. Also, $f''(x) = \dfrac{2x^2-12}{x^5}$ is positive (so f is CU) on $(-\sqrt{6}, 0)$ and $(\sqrt{6}, \infty)$, and negative (so f is CD) on $(-\infty, -\sqrt{6})$ and $(0, \sqrt{6})$. There are IP at $\left(\sqrt{6}, \frac{5\sqrt{6}}{36}\right)$ and $\left(-\sqrt{6}, -\frac{5\sqrt{6}}{36}\right)$.



14. $f(x) = \dfrac{\sqrt[3]{x}}{1-x} = x^{1/3}(1-x)^{-1} \quad \Rightarrow \quad f'(x) = x^{1/3}(-1)(1-x)^{-2}(-1) + (1-x)^{-1}\left(\frac{1}{3}\right)x^{-2/3} = \dfrac{x^{-2/3}}{3}\dfrac{1+2x}{(x-1)^2} \quad \Rightarrow$

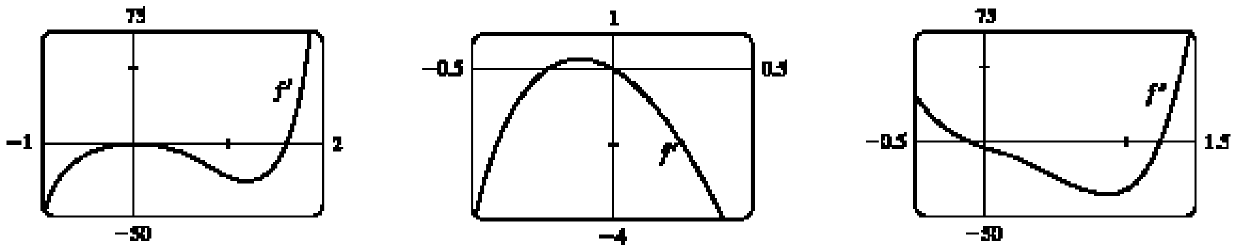
$$f''(x) = \frac{x^{-2/3}}{3}\frac{(x-1)^2(2)-(1+2x)(2)(x-1)}{(x-1)^4} + \frac{1+2x}{(x-1)^2}\left(\frac{-2x^{-5/3}}{9}\right) = -\frac{2x^{-5/3}}{9}\frac{5x^2+5x-1}{(x-1)^3}$$



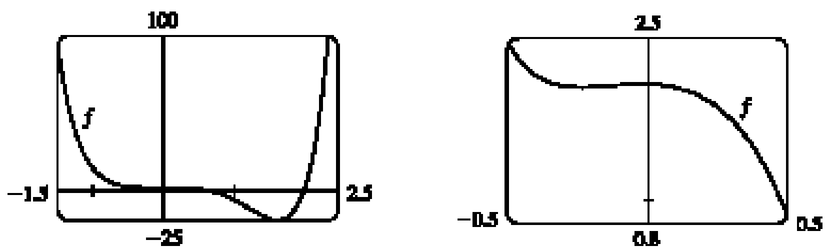
From the graphs, it appears that f is increasing on $(-0.50, 1)$ and $(1, \infty)$, with a vertical asymptote at $x = 1$, and decreasing on $(-\infty, -0.50)$; f has no local maximum, but a local minimum of about $f(-0.50) = -0.53$; f is CU on $(-1.17, 0)$ and $(0.17, 1)$ and CD on $(-\infty, -1.17)$, $(0, 0.17)$ and $(1, \infty)$; and f has inflection points at about $(-1.17, -0.49)$, $(0, 0)$ and $(0.17, 0.67)$. Note also that $\lim\limits_{x\to\pm\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote.

15. $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2 \quad \Rightarrow \quad f'(x) = 18x^5 - 25x^4 + 4x^3 - 15x^2 - 4x \quad \Rightarrow$

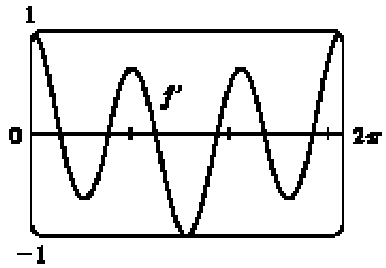
$f''(x) = 90x^4 - 100x^3 + 12x^2 - 30x - 4$

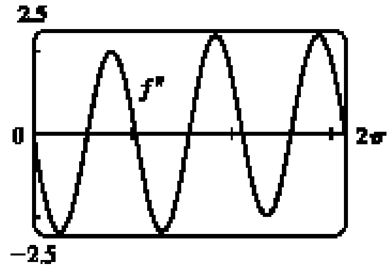


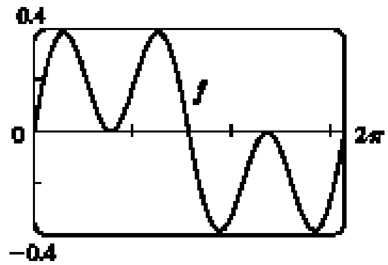
From the graphs of f' and f'', it appears that f is increasing on $(-0.23, 0)$ and $(1.62, \infty)$ and decreasing on $(-\infty, -0.23)$ and $(0, 1.62)$; f has a local maximum of about $f(0) = 2$ and local minima of about $f(-0.23) = 1.96$ and $f(1.62) = -19.2$; f is CU on $(-\infty, -0.12)$ and $(1.24, \infty)$ and CD on $(-0.12, 1.24)$; and f has inflection points at about $(-0.12, 1.98)$ and $(1.24, -12.1)$.



16. $f(x) = \sin x \cos^2 x \quad \Rightarrow \quad f'(x) = \cos^3 x - 2\sin^2 x \cos x \quad \Rightarrow \quad f''(x) = -7\sin x \cos^2 x + 2\sin^3 x$

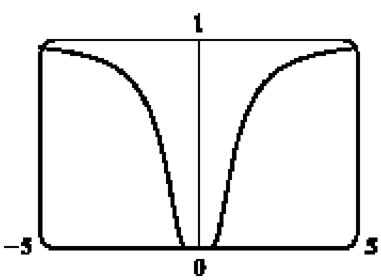






From the graphs of f' and f'', it appears that f is increasing on $(0, 0.62)$, $(1.57, 2.53)$, $(3.76, 4.71)$ and $(5.67, 2\pi)$ and decreasing on $(0.62, 1.57)$, $(2.53, 3.76)$ and $(4.71, 5.67)$; f has local maxima of about $f(0.62) = f(2.53) = 0.38$ and $f(4.71) = 0$ and local minima of about $f(1.57) = 0$ and $f(3.76) = f(5.67) = -0.38$; f is CU on $(1.08, 2.06)$, $(3.14, 4.22)$ and $(5.20, 2\pi)$ and CD on $(0, 1.08)$, $(2.06, 3.14)$ and $(4.22, 5.20)$; and f has inflection points at about $(0, 0)$, $(1.08, 0.20)$, $(2.06, 0.20)$, $(3.14, 0)$, $(4.22, -0.20)$, $(5.20, -0.20)$ and $(2\pi, 0)$.

17.



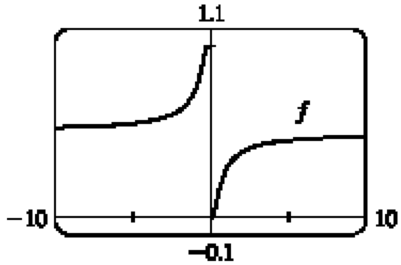
From the graph, we estimate the points of inflection to be about $(\pm 0.82, 0.22)$.

$f(x) = e^{-1/x^2} \quad \Rightarrow \quad f'(x) = 2x^{-3}e^{-1/x^2} \quad \Rightarrow$

$f''(x) = 2\left[x^{-3}(2x^{-3})e^{-1/x^2} + e^{-1/x^2}(-3x^{-4})\right] = 2x^{-6}e^{-1/x^2}\,(2 - 3x^2)$.

This is 0 when $2 - 3x^2 = 0 \quad \Leftrightarrow \quad x = \pm\sqrt{\frac{2}{3}}$, so the inflection points are $\left(\pm\sqrt{\frac{2}{3}}, e^{-3/2}\right)$.

18. (a)



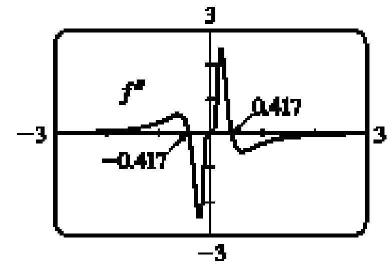
(b) $f(x) = \dfrac{1}{1+e^{1/x}}$. $\lim\limits_{x\to\infty} f(x) = \dfrac{1}{1+1} = \dfrac{1}{2}$,

$\lim\limits_{x\to-\infty} f(x) = \dfrac{1}{1+1} = \dfrac{1}{2}$,

$\lim\limits_{x\to0^+} f(x) = \dfrac{1}{1+\infty} = 0$, $\lim\limits_{x\to0^-} f(x) = \dfrac{1}{1+0} = 1$

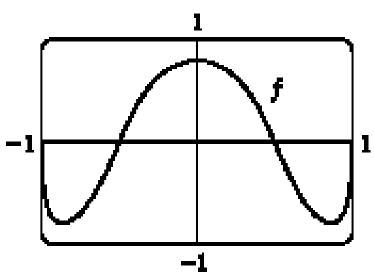
(c) From the graph of f, estimates for the IP are $(-0.4, 0.9)$ and $(0.4, 0.08)$.

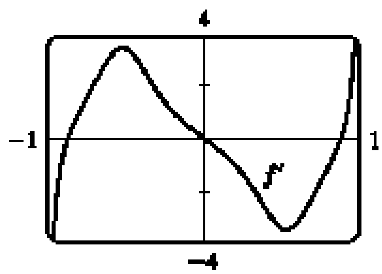
(d) $f''(x) = -\dfrac{e^{1/x}\left[e^{1/x}(2x-1)+2x+1\right]}{x^4(e^{1/x}+1)^3}$

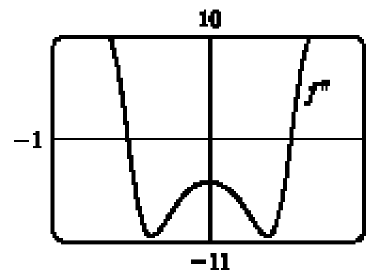


(e) From the graph, we see that f'' changes sign at $x = \pm 0.417$ ($x = 0$ is not in the domain of f). IP are approximately $(0.417, 0.083)$ and $(-0.417, 0.917)$.

19. $f(x) = \arctan(\cos(3\arcsin x))$. We use a CAS to compute f' and f'', and to graph f, f', and f'':







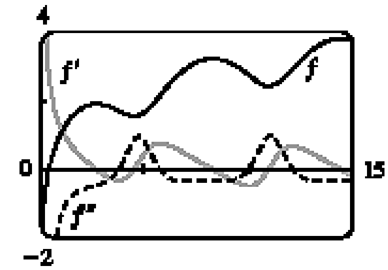
From the graph of f', it appears that the only maximum occurs at $x = 0$ and there are minima at $x = \pm 0.87$.

From the graph of f'', it appears that there are inflection points at $x = \pm 0.52$.

20. $f(x) = \ln(2x + x\sin x)$. We use the CAS to calculate

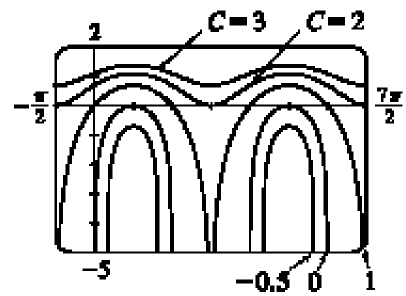
$$f'(x) = \frac{2 + \sin x + x\cos x}{2x + x\sin x} \text{ and}$$

$$f''(x) = \frac{2x^2 \sin x + 4\sin x - \cos^2 x + x^2 + 5}{x^2(\cos^2 x - 4\sin x - 5)}.$$



From the graphs, it seems that $f' > 0$ (and so f is increasing) on approximately the intervals $(0, 2.7)$, $(4.5, 8.2)$ and $(10.9, 14.3)$. It seems that f'' changes sign (indicating inflection points) at $x \approx 3.8, 5.7, 10.0$ and 12.0. Looking back at the graph of f, this implies that the inflection points have approximate coordinates $(3.8, 1.7)$, $(5.7, 2.1)$, $(10.0, 2.7)$, and $(12.0, 2.9)$.

21. The family of functions $f(x) = \ln(\sin x + C)$ all have the same period and all have maximum values at $x = \frac{\pi}{2} + 2\pi n$. Since the domain of ln is $(0, \infty)$, f has a graph only if $\sin x + C > 0$ somewhere. Since $-1 \le \sin x \le 1$, this happens if $C > -1$, that is, f has no graph if $C \le -1$. Similarly, if $C > 1$, then $\sin x + C > 0$ and f is continuous on $(-\infty, \infty)$. As C increases, the graph of f is shifted vertically upward and flattens out.



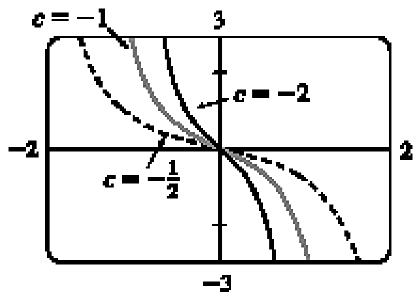
If $-1 < C \le 1$, f is defined where $\sin x + C > 0 \iff \sin x > -C \iff \sin^{-1}(-C) < x < \pi - \sin^{-1}(-C)$. Since the period is 2π, the domain of f is $\left(2n\pi + \sin^{-1}(-C), (2n+1)\pi - \sin^{-1}(-C)\right)$, n an integer.

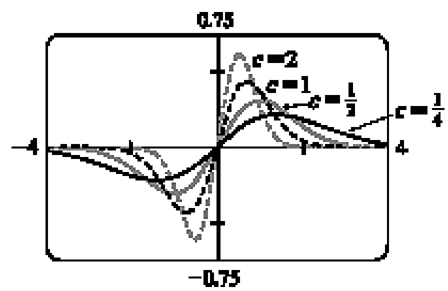
22. We exclude the case $c = 0$, since in that case $f(x) = 0$ for all x. To find the maxima and minima, we differentiate:

$$f(x) = cxe^{-cx^2} \quad \Rightarrow \quad f'(x) = c\left[xe^{-cx^2}(-2cx) + e^{-cx^2}(1)\right] = ce^{-cx^2}\left(-2cx^2 + 1\right).$$

This is 0 where $-2cx^2 + 1 = 0 \quad \Leftrightarrow \quad x = \pm 1/\sqrt{2c}$. So if $c > 0$, there are two maxima or minima, whose x-coordinates approach 0 as c increases. The negative root gives a minimum and the positive root gives a maximum, by the First Derivative Test. By substituting back into the equation, we see that $f\left(\pm 1/\sqrt{2c}\right) = c\left(\pm 1/\sqrt{2c}\right) e^{-c\left(\pm 1/\sqrt{2c}\right)^2} = \pm\sqrt{c/2e}$. So as c increases, the extreme points become more pronounced. Note that if $c > 0$, then $\lim\limits_{x\to\pm\infty} f(x) = 0$. If $c < 0$, then there are no extreme values, and $\lim\limits_{x\to\pm\infty} f(x) = \mp\infty$.

To find the points of inflection, we differentiate again: $f'(x) = ce^{-cx^2}\left(-2cx^2 + 1\right) \quad \Rightarrow$
$f''(x) = c\left[e^{-cx^2}(-4cx) + \left(-2cx^2 + 1\right)\left(-2cxe^{-cx^2}\right)\right] = -2c^2xe^{-cx^2}\left(3 - 2cx^2\right)$. This is 0 at $x = 0$ and where $3 - 2cx^2 = 0 \quad \Leftrightarrow \quad x = \pm\sqrt{3/(2c)} \quad \Rightarrow \quad$ IP at $\left(\pm\sqrt{3/(2c)}, \pm\sqrt{3c/2}\, e^{-3/2}\right)$. If $c > 0$ there are three inflection points, and as c increases, the x-coordinates of the nonzero inflection points approach 0. If $c < 0$, there is only one inflection point, the origin.





23. For $(1, 6)$ to be on the curve $y = x^3 + ax^2 + bx + 1$, we have that $6 = 1 + a + b + 1 \quad \Rightarrow \quad b = 4 - a$. Now $y' = 3x^2 + 2ax + b$ and $y'' = 6x + 2a$. Also, for $(1, 6)$ to be an inflection point it must be true that $y''(1) = 6(1) + 2a = 0 \quad \Rightarrow \quad a = -3 \quad \Rightarrow \quad b = 4 - (-3) = 7$. Note that with $a = -3$, we have $y'' = 6x - 6 = 6(x - 1)$, so y'' changes sign at $x = 1$, proving that $(1, 6)$ is a point of inflection. [This does not follow from the fact that $y''(1) = 0$.]

24. (a) $g(x) = f(x^2) \Rightarrow g'(x) = 2xf'(x^2)$ by the Chain Rule. Since $f'(x) > 0$ for all $x \neq 0$, we must have $f'(x^2) > 0$ for $x \neq 0$, so $g'(x) = 0 \Leftrightarrow x = 0$. Now $g'(x)$ changes sign (from negative to positive) at $x = 0$, since one of its factors, $f'(x^2)$, is positive for all x, and its other factor, $2x$, changes from negative to positive at this point, so by the First Derivative Test, f has a local and absolute minimum at $x = 0$.

(b) $g'(x) = 2xf'(x^2) \Rightarrow g''(x) = 2\left[xf''(x^2)(2x) + f'(x^2)\right] = 4x^2 f''(x^2) + 2f'(x^2)$ by the Product Rule and the Chain Rule. But $x^2 > 0$ for all $x \neq 0$, $f''(x^2) > 0$ (since f is CU for $x > 0$), and $f'(x^2) > 0$ for all $x \neq 0$, so since all of its factors are positive, $g''(x) > 0$ for $x \neq 0$. Whether $g''(0)$ is positive or 0 doesn't matter (since the sign of g'' does not change there); g is concave upward on $\mathbb{R}$.

25. $\displaystyle \lim_{x\to 0} \frac{\tan \pi x}{\ln(1+x)} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\pi \sec^2 \pi x}{1/(1+x)} = \frac{\pi \cdot 1^2}{1/1} = \pi$

26. $\displaystyle \lim_{x\to 0} \frac{1-\cos x}{x^2+x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\sin x}{2x+1} = \frac{0}{1} = 0$

27. $\displaystyle \lim_{x\to 0} \frac{e^{4x}-1-4x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0} \frac{4e^{4x}-4}{2x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{16e^{4x}}{2} = \lim_{x\to 0} 8e^{4x} = 8 \cdot 1 = 8$

28. $\displaystyle \lim_{x\to \infty} \frac{e^{4x}-1-4x}{x^2} \overset{\text{H}}{=} \lim_{x\to \infty} \frac{4e^{4x}-4}{2x} \overset{\text{H}}{=} \lim_{x\to \infty} \frac{16e^{4x}}{2} = \lim_{x\to \infty} 8e^{4x} = \infty$

29. $\displaystyle \lim_{x\to \infty} x^3 e^{-x} = \lim_{x\to \infty} \frac{x^3}{e^x} \overset{\text{H}}{=} \lim_{x\to \infty} \frac{3x^2}{e^x} \overset{\text{H}}{=} \lim_{x\to \infty} \frac{6x}{e^x} \overset{\text{H}}{=} \lim_{x\to \infty} \frac{6}{e^x} = 0$

30. $\displaystyle \lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} \left(-\tfrac{1}{2}x^2\right) = 0$

31. $\displaystyle \lim_{x\to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1^+} \left(\frac{x\ln x - x + 1}{(x-1)\ln x}\right) \overset{\text{H}}{=} \lim_{x\to 1^+} \frac{x\cdot(1/x) + \ln x - 1}{(x-1)\cdot(1/x) + \ln x}$

$\displaystyle = \lim_{x\to 1^+} \frac{\ln x}{1 - 1/x + \ln x} \overset{\text{H}}{=} \lim_{x\to 1^+} \frac{1/x}{1/x^2 + 1/x} = \frac{1}{1+1} = \frac{1}{2}$

32. $y = (\tan x)^{\cos x} \quad \Rightarrow \quad \ln y = \cos x \ln \tan x$, so

$$\lim_{x\to(\pi/2)^-} \ln y = \lim_{x\to(\pi/2)^-} \frac{\ln \tan x}{\sec x} \overset{\text{H}}{=} \lim_{x\to(\pi/2)^-} \frac{(1/\tan x)\sec^2 x}{\sec x \tan x} = \lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan^2 x}$$

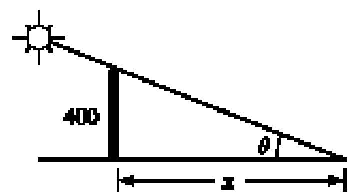
$$= \lim_{x\to(\pi/2)^-} \frac{\cos x}{\sin^2 x} = \frac{0}{1^2} = 0,$$

so $\lim_{x\to(\pi/2)^-} (\tan x)^{\cos x} = \lim_{x\to(\pi/2)^-} e^{\ln y} = e^0 = 1.$

33. We are given $d\theta/dt = -0.25$ rad/h. $\tan\theta = 400/x \quad \Rightarrow$

$x = 400\cot\theta \quad \Rightarrow \quad \dfrac{dx}{dt} = -400\csc^2\theta\,\dfrac{d\theta}{dt}$. When $\theta = \frac{\pi}{6}$,

$\dfrac{dx}{dt} = -400(2)^2(-0.25) = 400$ ft/h.

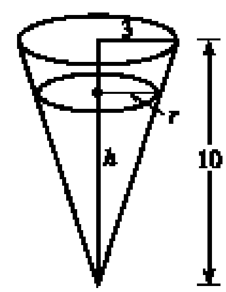


34. Given $dV/dt = 2$, find dh/dt when $h = 5$. $V = \frac{1}{3}\pi r^2 h$ and, from similar

triangles, $\dfrac{r}{h} = \dfrac{3}{10} \quad \Rightarrow \quad V = \dfrac{\pi}{3}\left(\dfrac{3h}{10}\right)^2 h = \dfrac{3\pi}{100}h^3$, so

$2 = \dfrac{dV}{dt} = \dfrac{9\pi}{100}h^2\,\dfrac{dh}{dt} \quad \Rightarrow \quad \dfrac{dh}{dt} = \dfrac{200}{9\pi h^2} = \dfrac{200}{9\pi\,(5)^2} = \dfrac{8}{9\pi}$ cm/s

when $h = 5$.

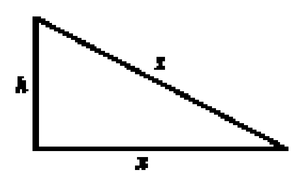


35. Given $dh/dt = 5$ and $dx/dt = 15$, find dz/dt. $z^2 = x^2 + h^2 \quad \Rightarrow$

$2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 2h\,\dfrac{dh}{dt} \quad \Rightarrow \quad \dfrac{dz}{dt} = \dfrac{1}{z}(15x + 5h)$. When $t = 3$,

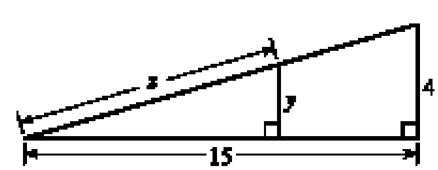
$h = 45 + 3(5) = 60$ and $x = 15(3) = 45 \quad \Rightarrow \quad z = \sqrt{45^2 + 60^2} = 75$,

so $\dfrac{dz}{dt} = \frac{1}{75}\,[15(45) + 5(60)] = 13$ ft/s.



36. We are given $dz/dt = 30$ ft/s. By similar triangles, $\dfrac{y}{z} = \dfrac{4}{\sqrt{241}} \quad \Rightarrow$

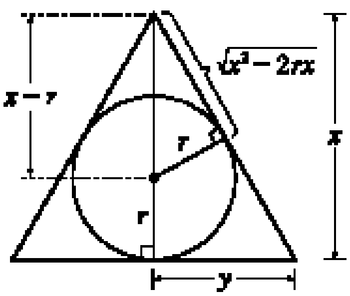
$y = \dfrac{4}{\sqrt{241}}z$, so $\dfrac{dy}{dt} = \dfrac{4}{\sqrt{241}}\,\dfrac{dz}{dt} = \dfrac{120}{\sqrt{241}} \approx 7.7$ ft/s.



37. Call the two integers x and y. Then $x + 4y = 1000$, so $x = 1000 - 4y$. Their product is $P = xy = (1000 - 4y)y$, so our problem is to maximize the function $P(y) = 1000y - 4y^2$, where $0 < y < 250$ and y is an integer. $P'(y) = 1000 - 8y$, so $P'(y) = 0 \quad \Leftrightarrow \quad y = 125$. $P''(y) = -8 < 0$, so $P(125) = 62{,}500$ is an absolute maximum. Since the optimal y turned out to be an integer, we have found the desired pair of numbers, namely $x = 1000 - 4(125) = 500$ and $y = 125$.

38. On the hyperbola $xy = 8$, if $d(x)$ is the distance from the point $(x, y) = (x, 8/x)$ to the point $(3, 0)$, then $[d(x)]^2 = (x - 3)^2 + 64/x^2 = f(x)$. $f'(x) = 2(x - 3) - 128/x^3 = 0 \quad \Rightarrow \quad x^4 - 3x^3 - 64 = 0 \quad \Rightarrow$ $(x - 4)\left(x^3 + x^2 + 4x + 16\right) = 0 \quad \Rightarrow \quad x = 4$ since the solution must have $x > 0$. Then $y = \frac{8}{4} = 2$, so the point is $(4, 2)$.

39.



By similar triangles, $\dfrac{y}{x} = \dfrac{r}{\sqrt{x^2 - 2rx}}$, so the area of the triangle is

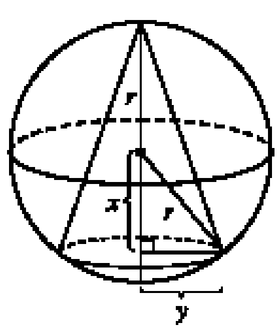
$$A(x) = \tfrac{1}{2}(2y)x = xy = \frac{rx^2}{\sqrt{x^2 - 2rx}} \quad \Rightarrow$$

$$A'(x) = \frac{2rx\sqrt{x^2 - 2rx} - rx^2(x - r)/\sqrt{x^2 - 2rx}}{x^2 - 2rx}$$

$$= \frac{rx^2\,(x - 3r)}{\left(x^2 - 2rx\right)^{3/2}} = 0 \ \text{ when } x = 3r.$$

$A'(x) < 0$ when $2r < x < 3r$, $A'(x) > 0$ when $x > 3r$. So $x = 3r$ gives a minimum and $A(3r) = r\left(9r^2\right)/\left(\sqrt{3}\,r\right) = 3\sqrt{3}\,r^2$.

40.



The volume of the cone is $V = \frac{1}{3}\pi y^2(r + x) = \frac{1}{3}\pi\left(r^2 - x^2\right)(r + x)$, $-r \le x \le r$.

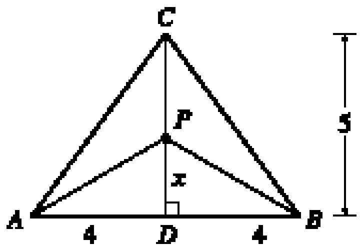
$$V'(x) = \tfrac{\pi}{3}\left[\left(r^2 - x^2\right)(1) + (r + x)(-2x)\right]$$

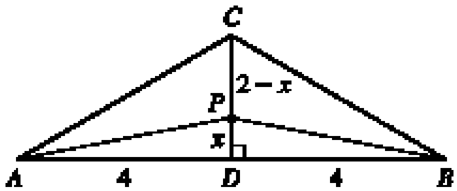
$$= \tfrac{\pi}{3}\left[(r + x)(r - x - 2x)\right] = \tfrac{\pi}{3}(r + x)(r - 3x)$$

$$= 0 \text{ when } x = -r \text{ or } x = r/3.$$

Now $V(r) = 0 = V(-r)$, so the maximum occurs at $x = r/3$ and the volume is $V\left(\dfrac{r}{3}\right) = \dfrac{\pi}{3}\left(r^2 - \dfrac{r^2}{9}\right)\left(\dfrac{4r}{3}\right) = \dfrac{32\pi r^3}{81}$.

41.

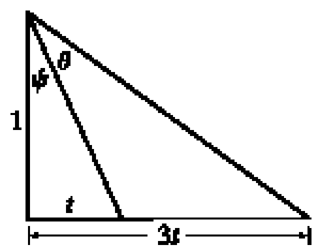




We minimize $L(x) = |PA| + |PB| + |PC| = 2\sqrt{x^2+16} + (5-x)$, $0 \le x \le 5$. $L'(x) = 2x/\sqrt{x^2+16} - 1 = 0 \quad\Leftrightarrow\quad 2x = \sqrt{x^2+16} \quad\Leftrightarrow$ $4x^2 = x^2 + 16 \quad\Leftrightarrow\quad x = \frac{4}{\sqrt{3}}$. $L(0) = 13$, $L\left(\frac{4}{\sqrt{3}}\right) \approx 11.9$, $L(5) \approx 12.8$, so the minimum occurs when $x = \frac{4}{\sqrt{3}} \approx 2.3$.

If $|CD| = 2$, $L(x)$ changes from $(5-x)$ to $(2-x)$ with $0 \le x \le 2$. But we still get $L'(x) = 0 \quad\Leftrightarrow\quad x = \frac{4}{\sqrt{3}}$, which isn't in the interval $[0, 2]$. Now $L(0) = 10$ and $L(2) = 2\sqrt{20} = 4\sqrt{5} \approx 8.9$. The minimum occurs when $P = C$.

42.



It suffices to maximize $\tan\theta$. Now

$$\frac{3t}{1} = \tan(\psi+\theta) = \frac{\tan\psi + \tan\theta}{1 - \tan\psi\,\tan\theta} = \frac{t + \tan\theta}{1 - t\tan\theta}.$$

So $3t(1 - t\tan\theta) = t + \tan\theta \quad\Rightarrow\quad 2t = \left(1 + 3t^2\right)\tan\theta \quad\Rightarrow$ $\tan\theta = \dfrac{2t}{1+3t^2}$. Let $f(t) = \tan\theta = \dfrac{2t}{1+3t^2} \quad\Rightarrow$

$$f'(t) = \frac{2\left(1+3t^2\right) - 2t(6t)}{\left(1+3t^2\right)^2} = \frac{2\left(1-3t^2\right)}{\left(1+3t^2\right)^2} = 0 \quad\Leftrightarrow\quad 1 - 3t^2 = 0 \quad\Leftrightarrow\quad t = \tfrac{1}{\sqrt{3}}$$

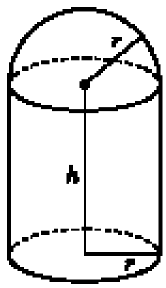
since $t \ge 0$. Now $f'(t) > 0$ for $0 \le t < \frac{1}{\sqrt{3}}$ and $f'(t) < 0$ for $t > \frac{1}{\sqrt{3}}$, so f has an absolute maximum when $t = \frac{1}{\sqrt{3}}$ and $\tan\theta = \dfrac{2\left(1/\sqrt{3}\right)}{1 + 3\left(1/\sqrt{3}\right)^2} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad \theta = \frac{\pi}{6}$. Substituting for t and θ in $3t = \tan(\psi+\theta)$ gives us $\sqrt{3} = \tan\left(\psi + \frac{\pi}{6}\right) \quad\Rightarrow\quad \psi = \frac{\pi}{6}$.

43. $v = K\sqrt{\dfrac{L}{C} + \dfrac{C}{L}} \quad\Rightarrow\quad \dfrac{dv}{dL} = \dfrac{K}{2\sqrt{(L/C)+(C/L)}}\left(\dfrac{1}{C} - \dfrac{C}{L^2}\right) = 0 \quad\Leftrightarrow\quad \dfrac{1}{C} = \dfrac{C}{L^2} \quad\Leftrightarrow\quad L^2 = C^2 \quad\Leftrightarrow\quad L = C.$

This gives the minimum velocity since $v' < 0$ for $0 < L < C$ and $v' > 0$ for $L > C$.

44.



We minimize the surface area $S = \pi r^2 + 2\pi rh + \frac{1}{2}(4\pi r^2) = 3\pi r^2 + 2\pi rh$.

Solving $V = \pi r^2 h + \frac{2}{3}\pi r^3$ for h, we get $h = \dfrac{V - \frac{2}{3}\pi r^3}{\pi r^2} = \dfrac{V}{\pi r^2} - \frac{2}{3}r$, so

$S(r) = 3\pi r^2 + 2\pi r\left[\dfrac{V}{\pi r^2} - \frac{2}{3}r\right] = \frac{5}{3}\pi r^2 + \dfrac{2V}{r}$.

$S'(r) = -\dfrac{2V}{r^2} + \frac{10}{3}\pi r = \dfrac{\frac{10}{3}\pi r^3 - 2V}{r^2} = 0 \quad\Leftrightarrow\quad \frac{10}{3}\pi r^3 = 2V \quad\Leftrightarrow$

$r^3 = \dfrac{3V}{5\pi} \quad\Leftrightarrow\quad r = \sqrt[3]{\dfrac{3V}{5\pi}}$. This gives an absolute minimum since $S'(r) < 0$ for $0 < r < \sqrt[3]{\dfrac{3V}{5\pi}}$ and $S'(r) > 0$

for $r > \sqrt[3]{\dfrac{3V}{5\pi}}$. Thus, $h = \dfrac{V - \frac{2}{3}\pi \cdot \frac{3V}{5\pi}}{\pi\sqrt[3]{\dfrac{(3V)^2}{(5\pi)^2}}} = \dfrac{\left(V - \frac{2}{5}V\right)\sqrt[3]{(5\pi)^2}}{\pi\sqrt[3]{(3V)^2}} = \dfrac{3V\sqrt[3]{(5\pi)^2}}{5\pi\sqrt[3]{(3V)^2}} = \sqrt[3]{\dfrac{3V}{5\pi}} = r$.

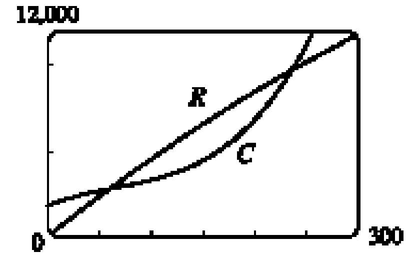
45. Let x denote the number of \$1 decreases in ticket price. Then the ticket price is $\$12 - \$1(x)$, and the average attendance is $11{,}000 + 1000(x)$. Now the revenue per game is

$$\begin{aligned} R(x) &= (\text{price per person}) \times (\text{number of people per game}) \\ &= (12 - x)(11{,}000 + 1000x) = -1000x^2 + 1000x + 132{,}000 \end{aligned}$$

for $0 \le x \le 4$ (since the seating capacity is 15,000) $\Rightarrow$ $R'(x) = -2000x + 1000 = 0 \quad\Leftrightarrow\quad x = 0.5$. This is a maximum since $R''(x) = -2000 < 0$ for all x. Now we must check the value of $R(x) = (12 - x)(11{,}000 + 1000x)$ at $x = 0.5$ and at the endpoints of the domain to see which value of x gives the maximum value of R. $R(0) = (12)(11{,}000) = 132{,}000$, $R(0.5) = (11.5)(11{,}500) = 132{,}250$, and $R(4) = (8)(15{,}000) = 120{,}000$. Thus, the maximum revenue of \$132,250 per game occurs when the average attendance is 11,500 and the ticket price is \$11.50.

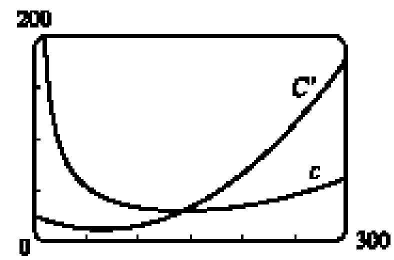
46. (a) $C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$ and $R(x) = xp(x) = 48.2x - 0.03x^2$. The profit is maximized when $C'(x) = R'(x)$.

From the figure, we estimate that the tangents are parallel when $x \approx 160$.

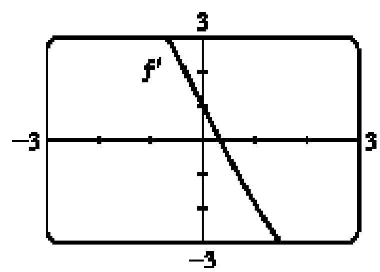


(b) $C'(x) = 25 - 0.4x + 0.003x^2$ and $R'(x) = 48.2 - 0.06x$. $C'(x) = R'(x) \Rightarrow 0.003x^2 - 0.34x - 23.2 = 0 \Rightarrow x_1 \approx 161.3$ $(x > 0)$. $R''(x) = -0.06$ and $C''(x) = -0.4 + 0.006x$, so $R''(x_1) = -0.06 < C''(x_1) \approx 0.57 \Rightarrow$ profit is maximized by producing 161 units.

(c) $c(x) = \dfrac{C(x)}{x} = \dfrac{1800}{x} + 25 - 0.2x + 0.001x^2$ is the average cost. Since the average cost is minimized when the marginal cost equals the average cost, we graph $c(x)$ and $C'(x)$ and estimate the point of intersection. From the figure, $C'(x) = c(x) \Leftrightarrow x \approx 144$.



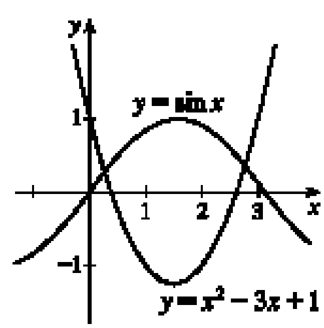
47. $f(t) = \cos t + t - t^2 \Rightarrow f'(t) = -\sin t + 1 - 2t$. $f'(t)$ exists for all t, so to find the maximum of f, we can examine the zeros of f'. From the graph of f', we see that a good choice for t_1 is $t_1 = 0.3$. Use $g(t) = -\sin t + 1 - 2t$ and $g'(t) = -\cos t - 2$ to obtain $t_2 \approx 0.33535293$, $t_3 \approx 0.33541803 \approx t_4$. Since $f''(t) = -\cos t - 2 < 0$ for all t, $f(0.33541803) \approx 1.16718557$ is the absolute maximum.



48. Graphing $y = \sin x$ and $y = x^2 - 3x + 1$ shows that there are two roots, one about 0.3 and the other about 2.8. $f(x) = \sin x - x^2 + 3x - 1 \Rightarrow$

$f'(x) = \cos x - 2x + 3 \Rightarrow x_{n+1} = x_n - \dfrac{\sin x_n - x_n^2 + 3x_n - 1}{\cos x_n - 2x_n + 3}$.

Now $x_1 = 0.3 \Rightarrow x_2 \approx 0.268552 \Rightarrow x_3 \approx 0.268881 \approx x_4$ and $x_1 = 2.8 \Rightarrow x_2 \approx 2.770354 \Rightarrow x_3 \approx 2.770058 \approx x_4$, so to six decimal places, the roots are 0.268881 and 2.770058.



49. $f(x) = e^x - (2/\sqrt{x}) = e^x - 2x^{-1/2} \quad \Rightarrow \quad F(x) = e^x - 2\dfrac{x^{-1/2+1}}{-1/2+1} + C = e^x - 2\dfrac{x^{1/2}}{1/2} + C = e^x - 4\sqrt{x} + C$

50. $g(t) = \dfrac{1+t}{\sqrt{t}} = \dfrac{1}{\sqrt{t}} + \dfrac{t}{\sqrt{t}} = t^{-1/2} + t^{1/2} \quad \Rightarrow \quad G(t) = 2t^{1/2} + \frac{2}{3}t^{3/2} + C$

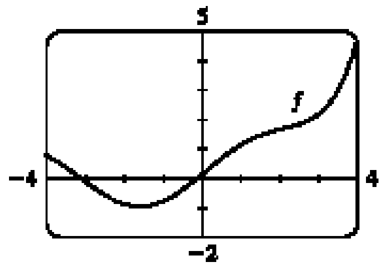
51. $f'(x) = 2/(1+x^2) \quad \Rightarrow \quad f(x) = 2\arctan x + C$. $f(0) = 2\arctan 0 + C = 0 + C = C$ and $f(0) = -1 \quad \Rightarrow \quad C = -1$.
Therefore, $f(x) = 2\arctan x - 1$.

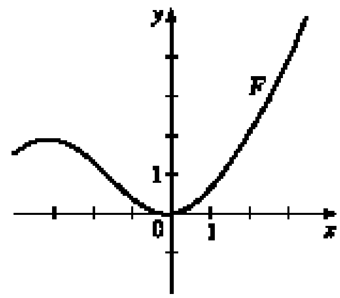
52. $f'(u) = \dfrac{u^2 + \sqrt{u}}{u} = u + u^{-1/2} \quad \Rightarrow \quad f(u) = \frac{1}{2}u^2 + 2u^{1/2} + C$.
$f(1) = \frac{1}{2} + 2 + C$ and $f(1) = 3 \quad \Rightarrow \quad C = \frac{1}{2}$, so $f(u) = \frac{1}{2}u^2 + 2\sqrt{u} + \frac{1}{2}$.

53. $f''(x) = 1 - 6x + 48x^2 \quad \Rightarrow \quad f'(x) = x - 3x^2 + 16x^3 + C$. $f'(0) = C$ and $f'(0) = 2 \quad \Rightarrow \quad C = 2$, so
$f'(x) = x - 3x^2 + 16x^3 + 2$ and hence, $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + D$.
$f(0) = D$ and $f(0) = 1 \quad \Rightarrow \quad D = 1$, so $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + 1$.

54. $f''(x) = 2x^3 + 3x^2 - 4x + 5 \quad \Rightarrow \quad f'(x) = \frac{1}{2}x^4 + x^3 - 2x^2 + 5x + C \quad \Rightarrow$
$f(x) = \frac{1}{10}x^5 + \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 + Cx + D$. $f(0) = D$ and $f(0) = 2 \quad \Rightarrow \quad D = 2$.
$f(1) = \frac{1}{10} + \frac{1}{4} - \frac{2}{3} + \frac{5}{2} + C + 2$ and $f(1) = 0 \quad \Rightarrow \quad C = -\frac{6}{60} - \frac{15}{60} + \frac{40}{60} - \frac{150}{60} - \frac{120}{60} = -\frac{251}{60}$, so
$f(x) = \frac{1}{10}x^5 + \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 - \frac{251}{60}x + 2$.

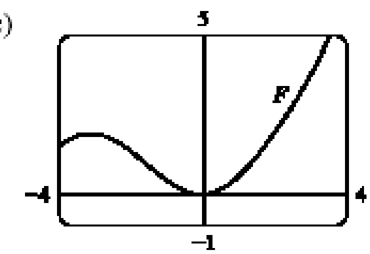
55. (a) Since f is 0 just to the left of the y-axis, we must have a minimum of F at the same place since we are increasing through $(0, 0)$ on F. There must be a local maximum to the left of $x = -3$, since f changes from positive to negative there.



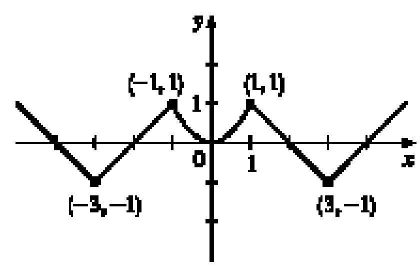


(b) $f(x) = 0.1e^x + \sin x \quad \Rightarrow$
$F(x) = 0.1e^x - \cos x + C$. $F(0) = 0 \quad \Rightarrow$
$0.1 - 1 + C = 0 \quad \Rightarrow \quad C = 0.9$, so
$F(x) = 0.1e^x - \cos x + 0.9$.

(c)



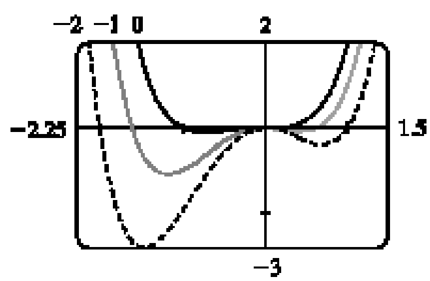
56. On $(0, 1)$, $f(x) = x^2$ since $f'(x) = 2x$ and $f(0) = 0$. On $(1, 3)$, f is linear with slope -1 since $f'(x) = -1$. If $x > 3$, f is linear with slope 1 since $f'(x) = 1$. Because f is an even function, we can just reflect this graph through the y-axis to get the complete graph.

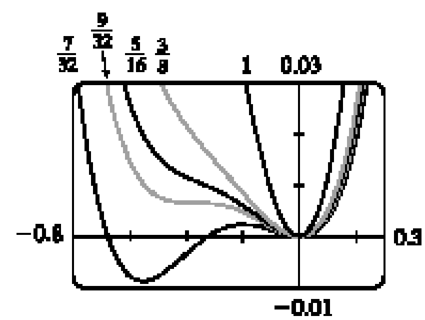


57. Choosing the positive direction to be upward, we have $a(t) = -9.8 \quad \Rightarrow \quad v(t) = -9.8t + v_0$, but $v(0) = 0 = v_0 \quad \Rightarrow$ $v(t) = -9.8t = s'(t) \quad \Rightarrow \quad s(t) = -4.9t^2 + s_0$, but $s(0) = s_0 = 500 \quad \Rightarrow \quad s(t) = -4.9t^2 + 500$. When $s = 0$, $-4.9t^2 + 500 = 0 \quad \Rightarrow \quad t_1 = \sqrt{\frac{500}{4.9}} \approx 10.1 \quad \Rightarrow \quad v(t_1) = -9.8\sqrt{\frac{500}{4.9}} \approx -98.995$ m/s. Since the canister has been designed to withstand an impact velocity of 100 m/s, the canister will *not burst*.

58. $f(x) = x^4 + x^3 + cx^2 \quad \Rightarrow \quad f'(x) = 4x^3 + 3x^2 + 2cx$. This is 0 when $x\left(4x^2 + 3x + 2c\right) = 0 \quad \Leftrightarrow \quad x = 0$ or $4x^2 + 3x + 2c = 0$. Using the quadratic formula, we find that the roots of this last equation are $x = \dfrac{-3 \pm \sqrt{9 - 32c}}{8}$. Now if $9 - 32c < 0 \quad \Leftrightarrow \quad c > \frac{9}{32}$, then $(0, 0)$ is the only critical point, a minimum. If $c = \frac{9}{32}$, then there are two critical points (a minimum at $x = 0$, and a horizontal tangent with no maximum or minimum at $x = -\frac{3}{8}$) and if $c < \frac{9}{32}$, then there are three critical points except when $c = 0$, in which case the root with the $+$ sign coincides with the critical point at $x = 0$. For $0 < c < \frac{9}{32}$, there is a minimum at $x = -\dfrac{3}{8} - \dfrac{\sqrt{9 - 32c}}{8}$, a maximum at $x = -\dfrac{3}{8} + \dfrac{\sqrt{9 - 32c}}{8}$, and a minimum at $x = 0$. For $c = 0$, there is a minimum at $x = -\frac{3}{4}$ and a horizontal tangent with no extremum at $x = 0$, and for $c < 0$, there is a maximum at $x = 0$, and there are minima at $x = -\dfrac{3}{8} \pm \dfrac{\sqrt{9 - 32c}}{8}$. Now we calculate $f''(x) = 12x^2 + 6x + 2c$. The roots of this equation are $x = \dfrac{-6 \pm \sqrt{36 - 4 \cdot 12 \cdot 2c}}{24}$. So if $36 - 96c \le 0 \quad \Leftrightarrow \quad c \ge \frac{3}{8}$, then there is no inflection point. If $c < \frac{3}{8}$, then there are two inflection points at $x = -\dfrac{1}{4} \pm \dfrac{\sqrt{9 - 24c}}{12}$.

Value of c	No. of CP	No. of IP
$c < 0$	3	2
$c = 0$	2	2
$0 < c < \frac{9}{32}$	3	2
$c = \frac{9}{32}$	2	2
$\frac{9}{32} < c < \frac{3}{8}$	1	2
$c \ge \frac{3}{8}$	1	0





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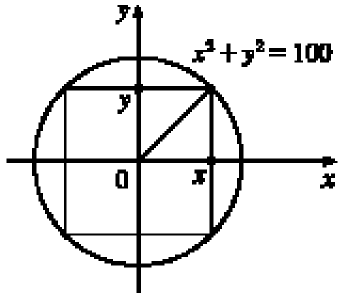
59. (a)



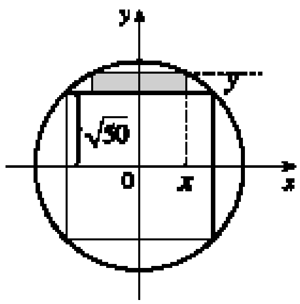
The cross-sectional area of the rectangular beam is $A = 2x \cdot 2y = 4xy = 4x\sqrt{100-x^2}$, $0 \le x \le 10$, so

$$\frac{dA}{dx} = 4x\left(\tfrac{1}{2}\right)\left(100-x^2\right)^{-1/2}(-2x) + \left(100-x^2\right)^{1/2} \cdot 4$$

$$= \frac{-4x^2}{(100-x^2)^{1/2}} + 4\left(100-x^2\right)^{1/2} = \frac{4\left[-x^2+\left(100-x^2\right)\right]}{(100-x^2)^{1/2}}.$$

$\frac{dA}{dx} = 0$ when $-x^2 + \left(100-x^2\right) = 0 \Rightarrow x^2 = 50 \Rightarrow x = \sqrt{50} \approx 7.07 \Rightarrow y = \sqrt{100 - \left(\sqrt{50}\right)^2} = \sqrt{50}$. Since $A(0) = A(10) = 0$, the rectangle of maximum area is a square.

(b)



The cross-sectional area of each rectangular plank (shaded in the figure) is $A = 2x\left(y - \sqrt{50}\right) = 2x\left[\sqrt{100-x^2} - \sqrt{50}\right]$, $0 \le x \le \sqrt{50}$, so

$$\frac{dA}{dx} = 2\left(\sqrt{100-x^2} - \sqrt{50}\right) + 2x\left(\tfrac{1}{2}\right)(100-x^2)^{-1/2}(-2x)$$

$$= 2(100-x^2)^{1/2} - 2\sqrt{50} - \frac{2x^2}{(100-x^2)^{1/2}}$$

Set $\frac{dA}{dx} = 0$: $(100-x^2) - \sqrt{50}\,(100-x^2)^{1/2} - x^2 = 0 \Rightarrow 100 - 2x^2 = \sqrt{50}\,(100-x^2)^{1/2} \Rightarrow$

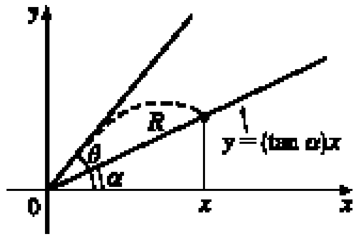
$10{,}000 - 400x^2 + 4x^4 = 50(100-x^2) \Rightarrow 4x^4 - 350x^2 + 5000 = 0 \Rightarrow$

$2x^4 - 175x^2 + 2500 = 0 \Rightarrow x^2 = \dfrac{175 \pm \sqrt{10{,}625}}{4} \approx 69.52$ or $17.98 \Rightarrow x \approx 8.34$ or 4.24.

But $8.34 > \sqrt{50}$, so $x_1 \approx 4.24 \Rightarrow y - \sqrt{50} = \sqrt{100 - x_1^2} - \sqrt{50} \approx 1.99$. Each plank should have dimensions about $8\frac{1}{2}$ inches by 2 inches.

(c) From the figure in part (a), the width is $2x$ and the depth is $2y$, so the strength is $S = k(2x)(2y)^2 = 8kxy^2 = 8kx\left(100-x^2\right) = 800kx - 8kx^3$, $0 \le x \le 10$. $dS/dx = 800k - 24kx^2 = 0$ when $24kx^2 = 800k \Rightarrow x^2 = \frac{100}{3} \Rightarrow x = \frac{10}{\sqrt{3}} \Rightarrow y = \sqrt{\frac{200}{3}} = \frac{10\sqrt{2}}{\sqrt{3}} = \sqrt{2}\,x$. Since $S(0) = S(10) = 0$, the maximum strength occurs when $x = \frac{10}{\sqrt{3}}$. The dimensions should be $\frac{20}{\sqrt{3}} \approx 11.55$ inches by $\frac{20\sqrt{2}}{\sqrt{3}} \approx 16.33$ inches.

60. (a)



$y = (\tan\theta)x - \dfrac{g}{2v^2\cos^2\theta}x^2$. The parabola intersects the line when

$(\tan\alpha)x = (\tan\theta)x - \dfrac{g}{2v^2\cos^2\theta}x^2 \quad\Rightarrow\quad x = \dfrac{(\tan\theta - \tan\alpha)2v^2\cos^2\theta}{g} \quad\Rightarrow$

$$R(\theta) = \frac{x}{\cos\alpha} = \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right)\frac{2v^2\cos^2\theta}{g\cos\alpha} = \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right)(\cos\theta\,\cos\alpha)\frac{2v^2\cos\theta}{g\cos^2\alpha}$$

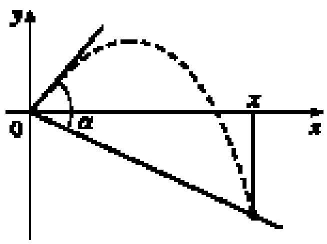
$$= (\sin\theta\,\cos\alpha - \sin\alpha\,\cos\theta)\frac{2v^2\cos\theta}{g\cos^2\alpha} = \sin(\theta-\alpha)\frac{2v^2\cos\theta}{g\cos^2\alpha}$$

(b) $R'(\theta) = \dfrac{2v^2}{g\cos^2\alpha}\left[\cos\theta\cdot\cos(\theta-\alpha) + \sin(\theta-\alpha)(-\sin\theta)\right] = \dfrac{2v^2}{g\cos^2\alpha}\cos[\theta + (\theta-\alpha)]$

$$= \frac{2v^2}{g\cos^2\alpha}\cos(2\theta-\alpha) = 0$$

when $\cos(2\theta-\alpha) = 0 \quad\Rightarrow\quad 2\theta - \alpha = \frac{\pi}{2} \quad\Rightarrow\quad \theta = \dfrac{\pi/2+\alpha}{2} = \dfrac{\pi}{4} + \dfrac{\alpha}{2}$. The First Derivative Test shows that this gives a maximum value for $R(\theta)$. [This could be done without calculus by applying the formula for $\sin x\cos y$ to $R(\theta)$.]

(c)



Replacing α by $-\alpha$ in part (a), we get $R(\theta) = \dfrac{2v^2\cos\theta\,\sin(\theta+\alpha)}{g\cos^2\alpha}$.

Proceeding as in part (b), or simply by replacing α by $-\alpha$ in the result of part (b), we see that $R(\theta)$ is maximized when $\theta = \dfrac{\pi}{4} - \dfrac{\alpha}{2}$.

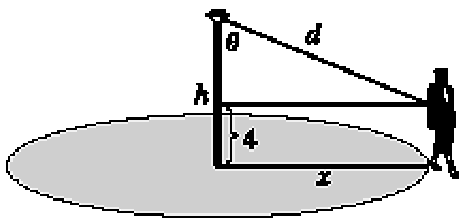
61. (a) $I = \dfrac{k\cos\theta}{d^2} = \dfrac{k(h/d)}{d^2} = k\dfrac{h}{d^3} = k\dfrac{h}{\left(\sqrt{40^2+h^2}\right)^3} = k\dfrac{h}{(1600+h^2)^{3/2}} \quad \Rightarrow$

$$\frac{dI}{dh} = k\frac{\left(1600+h^2\right)^{3/2} - h\frac{3}{2}\left(1600+h^2\right)^{1/2}\cdot 2h}{\left[\left(1600+h^2\right)^{3/2}\right]^2} = \frac{k\left(1600+h^2\right)^{1/2}\left(1600+h^2-3h^2\right)}{\left(1600+h^2\right)^3}$$

$$= \frac{k\left(1600-2h^2\right)}{\left(1600+h^2\right)^{5/2}} \quad [k \text{ is the constant of proportionality}]$$

Set $dI/dh = 0$: $1600 - 2h^2 = 0 \quad \Rightarrow \quad h^2 = 800 \quad \Rightarrow \quad h = \sqrt{800} = 20\sqrt{2}$. By the First Derivative Test, I has a local maximum at $h = 20\sqrt{2} \approx 28$ ft.

(b)



$\dfrac{dx}{dt} = 4$ ft/s

$$I = \frac{k\cos\theta}{d^2} = \frac{k[(h-4)/d]}{d^2} = \frac{k(h-4)}{d^3} = \frac{k(h-4)}{[(h-4)^2+x^2]^{3/2}} = k(h-4)\left[(h-4)^2+x^2\right]^{-3/2}$$

$$\frac{dI}{dt} = \frac{dI}{dx}\cdot\frac{dx}{dt} = k(h-4)\left(-\tfrac{3}{2}\right)\left[(h-4)^2+x^2\right]^{-5/2}\cdot 2x\cdot\frac{dx}{dt}$$

$$= k(h-4)(-3x)\left[(h-4)^2+x^2\right]^{-5/2}\cdot 4 = \frac{-12xk(h-4)}{[(h-4)^2+x^2]^{5/2}}$$

$$\left.\frac{dI}{dt}\right|_{x=40} = -\frac{480k(h-4)}{[(h-4)^2+1600]^{5/2}}$$

1. A function f has an **absolute maximum** at $x = c$ if $f(c)$ is the largest function value on the entire domain of f, whereas f has a **local maximum** at c if $f(c)$ is the largest function value when x is near c. See Figure 4 in Section 4.2.

2. (a) See Theorem 4.2.3.

(b) See the Closed Interval Method before Example 6 in Section 4.2.

3. (a) See Theorem 4.2.4.

(b) See Definition 4.2.5.

4. See the Mean Value Theorem in Section 4.3. Geometric interpretation—there is some point P on the graph of a function f [on the interval (a, b)] where the tangent line is parallel to the secant line that connects $(a, f(a))$ and $(b, f(b))$.

5. (a) See the I/D Test before Example 2 in Section 4.3.

(b) A function f is concave upward on an interval I if f' is an increasing function on I (or, equivalently, the graph of f lies above all of its tangent lines on I).

(c) See the Concavity Test before Example 4 in Section 4.3.

(d) An inflection point is a point where a curve changes its direction of concavity. They can be found by determining the points at which the second derivative changes sign.

6. (a) See the First Derivative Test after Example 2 in Section 4.3.

(b) See the Second Derivative Test before Example 4 in Section 4.3.

(c) See the note before Example 5 in Section 4.3.

7. (a) See l'Hospital's Rule and the three notes that follow it in Section 4.5.

(b) Write fg as $\dfrac{f}{1/g}$ or $\dfrac{g}{1/f}$.

(c) Convert the difference into a quotient using a common denominator, rationalizing, factoring, or some other method.

(d) Convert the power to a product by taking the natural logarithm of both sides of $y = f^g$ or by writing f^g as $e^{g \ln f}$.

8. Without calculus you could get misleading graphs that fail to show the most interesting features of a function. See the discussion following Figure 3 in Section 4.5 and the first paragraph in Section 4.4.

9. (a) See Figure 3 in Section 4.8.

(b) $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

(c) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

(d) Newton's method is likely to fail or to work very slowly when $f'(x_1)$ is close to 0.

10. (a) See the definition at the beginning of Section 4.9.

(b) If F_1 and F_2 are both antiderivatives of f on an interval I, then they differ by a constant.

1. False. For example, take $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(0) = 0$, but $f(0) = 0$ is not a maximum or minimum; $(0, 0)$ is an inflection point.

2. False. For example, $f(x) = |x|$ has an absolute minimum at 0, but $f'(0)$ does not exist.

3. False. For example, $f(x) = x$ is continuous on $(0, 1)$ but attains neither a maximum nor a minimum value on $(0, 1)$. Don't confuse this with f being continuous on the *closed* interval $[a, b]$, which would make the statement true.

4. True. By the Mean Value Theorem, $f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{0}{2} = 0$. Note that $|c| < 1 \quad \Leftrightarrow \quad c \in (-1, 1)$.

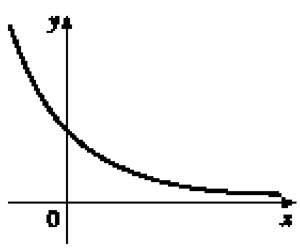
5. True. This is an example of part (b) of the I/D Test.

6. False. For example, the curve $y = f(x) = 1$ has no inflection points but $f''(c) = 0$ for all c.

7. False. $f'(x) = g'(x) \quad \Rightarrow \quad f(x) = g(x) + C$. For example, if $f(x) = x + 2$ and $g(x) = x + 1$, then $f'(x) = g'(x) = 1$, but $f(x) \neq g(x)$.

8. False. Assume there is a function f such that $f(1) = -2$ and $f(3) = 0$. Then by the Mean Value Theorem there exists a number $c \in (1, 3)$ such that $f'(c) = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{0 - (-2)}{2} = 1$. But $f'(x) > 1$ for all x, a contradiction.

9. True. The graph of one such function is sketched.



10. False. At any point $(a, f(a))$, we know that $f'(a) < 0$. So since the tangent line at $(a, f(a))$ is not horizontal, it must cross the x-axis—at $x = b$, say. But since $f''(x) > 0$ for all x, the graph of f must lie above all of its tangents; in particular, $f(b) > 0$. But this is a contradiction, since we are given that $f(x) < 0$ for all x.

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11. True. By the Mean Value Theorem, there exists a number c in $(0, 1)$ such that $f(1) - f(0) = f'(c)(1 - 0) = f'(c)$. Since $f'(c)$ is nonzero, $f(1) - f(0) \neq 0$, so $f(1) \neq f(0)$.

12. False. The most general antiderivative is $F(x) = -1/x + C_1$ for $x < 0$ and $F(x) = -1/x + C_2$ for $x > 0$ (see Example 1 in Section 4.9).

13. False. $\lim\limits_{x \to 0} \dfrac{x}{e^x} = \dfrac{\lim\limits_{x \to 0} x}{\lim\limits_{x \to 0} e^x} = \dfrac{0}{1} = 0$, not 1.