Math 1a
Review for Midterm II: Chapter 3

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Sections 3.1: Derivatives of Polynomials and Exponential Functions

Function	Derivative	Notes
x^n	nx^{n-1}	n is any real number
cf(x)	cf'(x)	c is a constant, f is differentiable
f(x) ± g(x)	f'(x) ± g'(x)	f and g are both differentiable
e^x	e^x	It's its own derivative!
a^x	$a^x\ln(a)$	More details about this later…

Notes:

- A **normal line** to a curve C at a point P is the line through P that is perpendicular to the tangent line at P
- The **tangent line** to a curve C = f(x) at a point P = (a, f(a)) is given by the formula y = f(a) + f'(a)(x-a)
- You should understand and be able to use the above derivatives
- You should understand and be able to apply the **definition of e**
  - **e is the number such that** $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$
- An even more useful limit to remember is the following, with $f(x) = a^x$
  - $\lim_{h\to 0} \frac{a^h - 1}{h} = f'(0)$ (re-read page 188 of the book to make sure you understand the proof)
  - You can just plug in a = e into the above limit to get the definition of e
- For **any** exponential function $f(x) = a^x$, the rate of change is proportional to the function itself
  - $f'(x) = a^x f'(0)$
- For $y = e^x$, the slope of the tangent line is equal to the y-coordinate of the point

Examples:

3.1.50: At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line 3x – y = 5?

3.1.56: Find the nth derivatives for *(a)* $f(x) = x^n$ and *(b)* f(x) = 1/x

A Weird Limit: Find $\lim_{x\to 0} \frac{10^x - 1}{x}$

Section 3.2: The Product and Quotient Rules

Function	Derivative	Notes
f(x)g(x)	f(x)g'(x) + g(x)f'(x)	f and g are both differentiable
$\frac{f(x)}{g(x)}$	$\frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$	f and g are both differentiable

Notes:

- There is a nice table of simple differentiation rules on page 198

Examples:

3.2.44: Find equations of the tangent lines to the curve y = (x-1)/(x+1) that are parallel to the line x – 2y = 2

3.2.46: If F(x) = f(x)g(x) where f and g have derivatives of all orders, find F''(x).

Section 3.3: Rates of Change in the Natural and Social Sciences

Notes:

- The **average rate of change** between two points is the slope of the secant line between the two points
- The **instantaneous rate of change** at a point is the slope of the tangent line at that point
- This section consists mainly of interpreting word problems

Examples:

Rabbits: Suppose a rabbit population starts with 1000 rabbits and doubles every year. *(a)* What is the population after 1 year? 2 years? 3 years? After n years? *(b)* What is the rate of increase of rabbits after 20 years? (This problem is a lot like 3.3.24 in the textbook)

Section 3.4: Derivatives of Trigonometric Functions

Function	Derivative	Notes
sinx	cosx	Remember how these graphs look!
cosx	- sinx	
tanx	$\sec^2 x$	
cscx	-cscxcotx	
secx	secxtanx	
cotx	$-\csc^2 x$	

Notes:

- **Important limits:**
  - $\lim_{x\to 0} \frac{\sin x}{x} = 1$
  - $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$
- **Some Helpful Trig Identities:**
  - cscx = 1/sinx
  - secx = 1/cosx
  - tanx = sinx/cosx
  - cotx = 1/tanx
  - sin2x = 2sinxcosx
- Remember that $\sin^2 x$ means $(\sin x)^2$, but $\sin^{-1} x$ usually means arcsinx and NOT 1/sinx
  - If the notation is unclear, ask!

Examples:

3.4.12: Differentiate y = cscx(x + cotx)

3.4.28: Find the points on the curve y = (cosx)/(2+sinx) at which the tangent is horizontal.

A Proof: Use the derivatives of sinx and cosx to find the derivative of tanx.

Section 3.5: The Chain Rule

<u>**Notes:**</u>

- **The Chain Rule**
  - If g is differentiable at x and f is differentiable at g(x), then the composite function F(x) = f(g(x)) is differentiable at x and F' is given by the product:
    - **F'(x) = f'(g(x))·g'(x)**
- Don't worry about the section on tangents to parametric curves

<u>**Examples:**</u>

a^x revisited: Using the chain rule and what we know about e^x to show that the derivative of a^x is $a^x \ln a$

The Absolute Value Function (Know this!): Write $|x| = (x^2)^{1/2}$ and show that the derivative if |x| is x/|x|

3.5.80: If f(x) = |sinx|, find f'(x). Where is f not differentiable?

3.5.26: Find the derivative of $y = (e^u - e^{-u})/(e^u + e^{-u})$

3.5.50 Suppose f is differentiable everywhere, and *a* is a real number. Let $F(x) = f(x^a)$, and $G(x) = [f(x)]^a$. Find expressions for F'(x) and G'(x).

Differentiate This: $f(x) = \cos(e^{\tan(\sin(5x)) + 2})$

Section 3.6: Implicit Differentiation

Notes:

- Some functions are defined implicitly by a relation between x and y:
  - $f(y) = g(x)$
- We don't need to solve an equation for y in terms of x in order to find the derivative of y. Instead, we differentiate both sides of the equation implicitly, and then solve for y'
  - $f'(y) \cdot y' = g'(x)$, by the chain rule
- Two curves are called **orthogonal** if at each point of intersection their tangent lines are perpendicular
  - This means that at points of intersection, the derivatives of the two curves are negative reciprocals of each-other
  - In other words, at points of intersection, multiplying the derivatives of the two curves will give -1
- Two families of curves are called **orthogonal trajectories** of each-other if every curve in one family is orthogonal to every curve in the other family
- We can use implicit differentiation to find the derivatives of inverse trigonometric functions
- In fact, we can use implicit differentiation to find the derivative of ANY inverse function

Function	**Derivative**	**Notes**
$\sin^{-1}x$	$1/(1-x^2)^{1/2}$	See page 237
$\cos^{-1}x$	$-1/(1-x^2)^{1/2}$	See below
$\tan^{-1}x$	$1/(1+x^2)$	See page 237
$f^{-1}(x)$	$1/f'(f^{-1}(x))$	Only works if f is a one-to-one differentiable function, and denominator isn't 0

Examples

3.6.36: The inverse cosine function is defined as the inverse of the restricted cosine function $f(x) = \cos x$, $0 \le x \le \pi$. Therefore, $y = \cos^{-1}x$ means that $\cos y = x$, and $0 \le y \le \pi$. Use implicit differentiation to find the derivative of $\cos^{-1}x$.

3.6.46: Show that these two families of curves are orthogonal trajectories: $y = ax^3$, $x^2 + 3y^2 = b$

3.6.52: Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point (12,3).

Section 3.7: Derivatives of Logarithmic Functions

Function	Derivative	Notes
$\log_a(x)$	$1/(x\ln a)$	See proof below in examples
$\ln x$	$1/x$	
$\ln[g(x)]$	$g'(x)/g(x)$	g needs to be differentiable, $g(x)\neq 0$
$\ln\lvert x\rvert$	$1/x$	

Notes:

- **Steps in Logarithmic Differentiation**
  - Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify (review your precalc!)
  - Differentiate implicitly with respect to x
  - Solve the resulting equation for y'
- **The number e as a limit**
  - $e = \lim_{x\to 0} (1+x)^{1/x}$
    $e = \lim_{n\to\infty} (1 + 1/n)^n$

Examples:

Differentiate this: $f(x) = [\sin(e^x)]\cdot[\ln x]$

Differentiate this: Use logarithmic differentiation to find the derivative of $y = \sin x^{\cos x}$

3.7.42 Show that $\lim_{n\to\infty} (1+x/n)^n = e^x$ for any $x>0$

Section 3.8: Linear Approximations and Differentials

Notes:

- The **linear approximation, tangent line approximation,** or **linearization** of f at a is given by:
  - o L(x) = f(a) + f'(a)(x-a)
- If y = f(x), where f is a differentiable function, the **differential dx** can take any value, and the **differential dy** is given by:
  - o dy = f'(x)dx
- **Relative Error** = Error/Measurement, approximated using dy/y
- **Percentage Error** = 100%(Relative Error)

Examples:

3.8.18 Use a linear approximation (or differentials) to estimate 1/1002

3.8.28 The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm. Use differentials to estimate the maximum error in the calculated area of the disk. What are the relative and percentage errors?

3.8.30 Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m