# Math 1a
# Review for Midterm II: Chapter 4

Course Assistant: Nicole Ali

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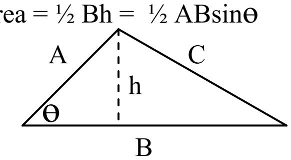
## Section 4.1: Related Rates

**Notes:**

- **Strategy:**
  - Read the problem, draw a picture
  - Name the variables
  - Write down rates of change, known quantities
  - Write down an equation relating the variables
  - Differentiate with respect to time
  - Substitute known quantities
  - Solve for unknown variables
- **Helpful Formulae:**
  - Area = ½ Bh = ½ ABsinθ $\qquad C^2 = A^2 + B^2 - 2AB\cos\theta$



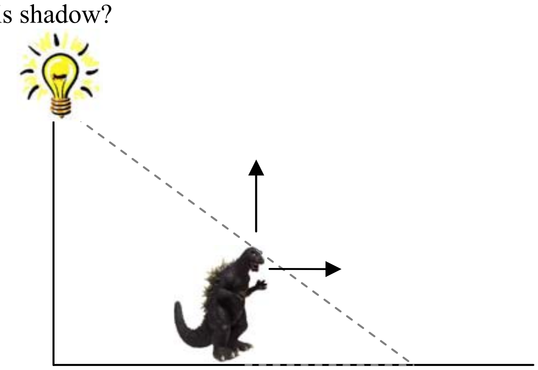

**Examples:**
**From one of our handouts:**

4. Scooby Doo and Shaggy are in the woods. Fred told them to look for monsters, but they're really looking for a snack. They come to a clearing, and out of the woods steps a wolfman! Shaggy tries to hide by climbing up a very tall tree at a rate of 4 ft/s. Ten feet from the base of the tree, Scooby tries to hide by digging a hole straight down at a rate of 3 ft/s. At what rate is the distance between Scooby and Shaggy changing after 10 seconds?

**Examples (Continued)**
**Also from a previous exam:**

(2) Show that for a given fixed perimeter the isosceles triangle (two sides of equal length) that gives the most area is an equilateral triangle (three sides of equal length).

**The Godzilla Problem:** When Godzilla is standing at a 100 feet tall lamp-post, Godzilla is 10 feet tall. Godzilla then starts walking away from the lamp-post at a speed of 10 ft/sec, and at the same time he starts walking, Godzilla starts growing upwards at a rate of 1 f/sec. When Godzilla is 50 feet tall, what is the rate of change of the length of his shadow?



## Section 4.2: Maximum and Minimum Values

**Notes:**

- f has an **absolute maximum (global maximum)** at c if $f(c) \geq f(x)$ for all x in the domain of f.
- f has an **absolute minimum (global minimum)** at c if $f(c) \leq f(x)$ for all x in the domain of f.
- The maximum and minimum values of f are called the **extreme values** of f
- f has a **local maximum (relative maximum)** at c if $f(c) \geq f(x)$ when x is near c
- f has a **local minimum (relative minimum)** at c if $f(c) \leq f(x)$ when x is near c
- Note that an absolute max/min is sometimes also a local max/min
- **The Extreme Value Theorem**
    - If f is continuous on a closed interval [a,b], then f attains an absolute maximum value f(c) and an absolute minimum value f(d) at some numbers c and d in [a,b]
- **Fermat's Theorem**
    - If f has a local maximum or minimum at c, and if f'(c) exists, then f'(c) = 0
    - Alternatively, if f has a local max or min at c, then c is a critical number of f
- A **critical number** of a function f is a number c in the domain of f such that either f'(c) = 0 or f'(c) doesn't exist
- **The Closed Interval Method**
    - To find the absolute max and min values of a continuous function f on a closed interval [a,b]:
        - Find the values of f at the critical numbers of f in (a,b)
        - Find the values of f at the endpoints of the intervals
        - See which values are largest/smallest

**Examples:**

**4.2.28:** Find the critical numbers of the function $g(t) = |3t - 4|$

**4.2.48:** Find the absolute max and min values of $f(x) = x - \ln x$ on the interval [1/2, 2]

## Section 4.3: Derivatives and the Shapes of Curves

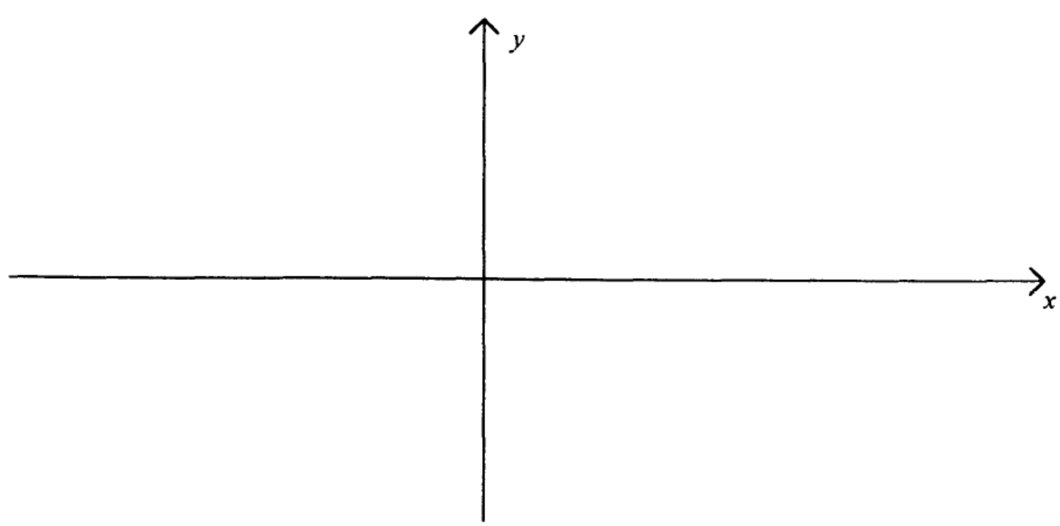
| **f** | **f'** | **f''** |
|---|---|---|
| Increasing | Greater than 0 | |
| Decreasing | Less than 0 | |
| Min/Max | Equals 0 | |
| Concave up | Increasing | Greater than 0 |
| Concave down | Decreasing | Less than 0 |

**Notes:**

- **The Mean Value Theorem**
  - If f is differentiable on [a,b], then there exists a number c between a and b such that:
    - $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
  - The slope of the secant line between a and b is equal to the slope of the tangent line at some point c between a and b
  - The average velocity over an interval equals the instantaneous velocity at some point c
- If f'(c) = 0, we use the first or second derivative test to see if c is a max, a min, or nothing
  - **First Derivative Test:**
    - If f' changes from negative to positive, then c is at a min
    - If f' changes from positive to negative, then c is at a max
  - **Second Derivative Test:**
    - If f''(c) > 0, then f has a min at c
    - If f''(c) < 0, then f has a max at c

**Examples:**
**From a previous exam:**

(3) (a) Sketch the graph of the function $f(x) = \dfrac{x^2 + 5x + 4}{x}$ showing intercepts, critical points, local maxima and minima, points of inflection, and asymptotes.



**Examples Continued:**

**4.3.54:** Suppose that $3 \leq f'(x) \leq 5$ for all values of x. Show that $18 \leq f(8) - f(2) \leq 30$.

**From a Previous Exam:**

**2.** (8 Points) Let $f$ be the function $f(x) = \sin(x) - \frac{x}{3}$.

According to the intermediate value theorem, there is a point c in [π/2, π] such that f(c) = 0

(b) Suppose that there was another point $d$ in the same interval such that $f(d) = 0$. What fact would the Mean Value Theorem allow you to conclude? Why is this "fact" *impossible*?

Once you have shown this, you will have proven that $c$ is the unique solution in the interval to the equation $f(x) = 0$.

**4.4.57** Show that a cubic function always has exactly one point of inflection.

## Section 4.4: We Skipped This. Yay!

## Section 4.5: Indeterminate Forms and l'Hospital's Rule

**Notes:**

- **Indeterminate forms**: $0/0$, $\infty/\infty$, $0\cdot\infty$, $\infty - \infty$, $0^0$, $\infty^0$, $1^\infty$
- **L'Hospital's Rule (for type 0/0 or ∞/∞):**
  - Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a), and we have an indeterminate form of type $0/0$ or $\infty/\infty$. Then:
    - $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$
- **Dealing with type 0·∞**
  - Re-write the product as a quotient and use L'Hospital's Rule
- **Dealing with type** $\infty - \infty$
  - Try to re-write it as a quotient using rationalization, common denominator, or factoring
- **Dealing with type** $0^0$, $\infty^0$, or $1^\infty$
  - Take the natural log of the limit L
  - Use L'Hospital's rule to find ln L
  - Remember that $L = e^{\ln L}$

**Examples:**

**4.5.46:** Use L'hospital's Rule to help fine the asymptotes of $f(x) = e^x/x$

**From Previous Exams:** Find the following limits.

$$\lim_{x\to 0^+} (\cos x)^{1/x}$$

$$\lim_{x\to 2} \frac{x^2-4}{x^2+4}$$

$$\lim_{x\to 1} \frac{x-1}{e^x-1}$$

# Section 4.6: Optimization Problems

**Notes:**

- **Basic Strategy:**
  - Draw a diagram
  - Introduce notation
  - Write down a function f for what you want to maximize or minimize
  - Write down the domain of f
  - Use calculus to find the absolute maximums or minimums (usually using the Closed Interval Method)
- Remember the first and second derivative tests!
- Remember to check the endpoints of your interval!

**Examples:**

**From a Previous Exam:**

**3.** (15 Points) A fish tank is to be designed with square base and rectangular sides (and no top). The material for the base costs five times as much as the glass for the sides. What are the dimensions of the tank which has volume $20\text{ft}^3$ and costs the least to build?

**A numbers problem:** Find two positive integers such that the sum of the first number and twice the second number is 30 and the product of the numbers is as large as possible.

**4.6.30** A boat leaves a dock at 2:00 pm and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 pm. At what time were the two boats closest together?

# Section 4.7: Applications to Business and Economics

**Notes:**

- **Cost function**: C(x) = cost of producing x units of a certain product
- **Marginal cost**: C'(x) = rate of change of C with respect to x
- **Average cost function**: c(x) = C(x)/x = cost per unit when x units is produced
    - o If the **average cost is minimum**, then C'(x) = c(x) = C(x)/x
- **Demand function**: p(x) = price per unit that the company can charge if it sells x units
- **Revenue function**: R(x) = xp(x) = sales function = total revenue
- **Marginal revenue function**: R'(x)
- **Profit function**: P(x) = total profit = R(x) – C(x)
- **Marginal profit function**: P'(x)
    - o If the **profit is maximum**, then R'(x) = C'(x) AND R''(x) < C''(x)

**Examples:**

**From a Previous Exam:**

**7.** (18 Points) Apex Corporation is planning to sell sponges on television. The sponges cost \$2 per package. Stacy believes that if they set the selling price at \$20 per package, they will sell 1000 packages, and for every dollar they increase the price, the quantity they will sell will decrease by 50 packages.

We will find the price at which profit will be maximized.

(a) Assuming Stacy's assumptions about the market are true, show that the demand curve (price in terms of quantity sold) is given by

$$p(x) = 40 - \tfrac{1}{50}x.$$

How many packages will be sold if the price is set at \$27?

(b) Now show that the profit (this is revenue minus costs, remember) is given by

$$K(x) = -\tfrac{1}{50}x(x - 1900)$$

(c) What price maximizes profit? Make sure you show it's maximal and not minimal!

# Section 4.8: Newton's Method (Newton-Raphson Algorithm)

**Notes:**

- We use this method to find the roots of functions
- A "root" of a function is where it is equal to zero, ie x is a root if $f(x) = 0$
- To find a root of $f(x)$ using Newton's method, we first guess a value $x_1$, and then we can make better and better guesses ($x_2$, $x_3$, $x_4$, …) using the following method:
  - $x_{n+1} = x_n - f(x_n)/f'(x_n)$

**Examples:**

**4.8.21**: Apply Newton's method to the equation $x^2 - a = 0$ to derive the following square-root algorithm:

$$x_{n+1} = ½ (x^n + a/x^n)$$

**A Cube Root Algorithm:** Use Newton's method to come up with an algorithm for finding the fourth roots of a number b. (Find a formula for $x_{n+1}$)

**4.8.9** Set up the equations you would use if you wanted to use Newton's method to find $1000^{1/7}$

That's the end of it! Now relax, try out one of the old midterms, and get some sleep!