Math 1a: Section 10/13/05
Nicole's Notes

Topic: Review of Limits and Continuity

Definition of a Limit

We write $\lim_{x \to a} f(x) = L$ if we can make the values of f(x) arbitrarily close to L by taking x to be sufficiently close to *a* (on both sides) but not equal to *a*

One-Sided Limits

Left-Hand Limit
We write $\lim_{x \to a^-} f(x) = L$ if we can make the values of f(x) arbitrarily close to L by taking x to be sufficiently close to *a* and x less than *a*

Right-Hand Limit
We write $\lim_{x \to a^+} f(x) = L$ if we can make the values of f(x) arbitrarily close to L by taking x to be sufficiently close to *a* and x greater than *a*

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

The Limit Laws

Suppose that c is a constant and the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} [f(x) \div g(x)] = \lim_{x \to a} f(x) \div \lim_{x \to a} g(x)$ if $\lim_{x \to a} g(x) \neq 0$
6. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$ where n is a positive integer, or n = 1/(positive integer)
7. $\lim_{x \to a} c = c$
8. $\lim_{x \to a} x = a$

More Rules Concerning Limits

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ provided the limits exist.

Theorem: If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

The Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$

Definition of Continuity

A function f is **continuous at a number *a*** if $\lim_{x \to a} f(x) = f(a)$

f is **continuous from the right at a number *a*** if $\lim_{x \to a+} f(x) = f(a)$

f is **continuous from the left at a number *a*** if $\lim_{x \to a-} f(x) = f(a)$

A function is **continuous on an interval** if it is continuous at every number in the interval.

Theorem: If f and g are continuous at *a* and *c* is a constant, the following functions are also continuous at a: (1) $f + g$ (2) $f - g$ (3) cf (4) fg (5) f/g if $g(a) \neq 0$

Theorem: Any polynomial is continuous everywhere. Polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, and logarithmic functions are continuous at every number in their domains (everywhere they are defined.

Theorem: If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$

Theorem: If g is continuous at *a* and f is continuous at g(*a*), then the composite function $f \circ g$ given my $(f \circ g)(x) = f(g(x))$ is continuous at *a*

The Intermediate Value Theorem: Suppose that f is continuous on the closed interval [a,b] and let N be any number between f(a) and f(b), where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$.

Limits Involving Infinity

Note: If the limit of a function is "infinity," the an actual limit does not actually exist. On tests and stuff, answer that the limit is infinity, but to be safe, you might want to write (DNE) as well

The line x = a is called a **vertical asymptote** of the curve y = f(x) if at least one of the following statements is true:

$\lim_{x\to a} f(x) = \infty$ $\lim_{x\to a^-} f(x) = \infty$ $\lim_{x\to a^+} f(x) = \infty$

$\lim_{x\to a} f(x) = -\infty$ $\lim_{x\to a^-} f(x) = -\infty$ $\lim_{x\to a^+} f(x) = -\infty$

Limits at infinity: Let f be a function defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ means that the values of f(x) can be made as close to L as we like by taking x sufficiently large.

The line y = L is called a **horizontal asymptote** of the curve y = f(x) if either:

$\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$

If n is a positive integer, then $\lim_{x\to\infty} 1/x^n = 0$ and $\lim_{x\to-\infty} 1/x^n = 0$

NOTE: $\infty - \infty$ is NOT zero, since ∞ isn't actually a number

Examples of limits where $x \to \pm\infty$:

① If the numerator and denominator have the same degree, the limit is the ratio of the leading coefficients

$$\lim_{x\to\infty} \frac{2x+3}{5x+7} = \lim_{x\to\infty} \frac{\frac{2x}{x} + \frac{3}{x}}{\frac{5x}{x} + \frac{3}{x}} = \frac{2+0}{5+0} = \frac{2}{5}$$

② If the degree of the numerator is less than the degree of the denominator, then the limit is zero

$$\lim_{x\to\infty} \frac{1}{x^3-4x+1} = \lim_{x\to\infty} \frac{\frac{1}{x^3}}{\frac{x^3}{x^3} - \frac{4x}{x^3} + \frac{1}{x^3}} = \frac{0}{1-0+0} = 0$$

③ If the degree of the numerator is greater than the degree of the denominator, the limit is infinity (DNE)

$$\lim_{x\to\infty} \frac{3x^2-6x}{4x-8} = \lim_{x\to\infty} \frac{\frac{3x^2}{x} - \frac{6x}{x}}{\frac{4x}{x} - \frac{8}{x}} = \lim_{x\to\infty} \frac{3x-6}{4-0} = \infty$$