Iain and Jean's Midterm 2 Study Notes
December 11, 2005

Things to Know

I. The Basics
   a. Trig identities, properties of logarithms
   b. Derivatives
      i. Logarithmic differentiation
      ii. Implicit differentiation
   c. Graphs
      i. Asymptotes/continuity
      ii. increasing/decreasing
      iii. concavity
      iv. relative extrema
II. Mean Value Theorem
   a. Related: Extreme Value Theorem, Fermat's Theorem
III. L'Hospital's Rule
   a. Interdeterminate vs. determinate forms
IV. Applications
   a. Linearization – equations of tangent lines
   b. Related rates
      i. Strategies
         1. Draw a diagram.
         2. Figure out what is changing with respect to what else
         3. Find a way to relate the variables
         4. Differentiate and solve
   c. Optimization
      i. Draw a diagram
      ii. Figure out what you are optimizing and what you are changing
      iii. Find a way to relate the variables
      iv. Differentiate and solve

Wrap-up: What to Expect (Breakdown of types of problems to know)

- Differentiation (a few problems) – make sure you know how to use and apply all rules, know when to use logarithmic differentiation (even though they'll probably tell you)
- Implicit Differentiation – be able to use it flexibly, don't be scared of graphs
- Graphs (increasing/decreasing, concavity, etc.)
- Mean Value Theorem – know it well! (very underrated)
- L'Hospital (a few problems)
- Related rates
- Optimization

Tips for Studying

- Make yourself a study sheet that includes formulas, statements of important theorems, and examples of special graphs, etc.
- Don't psych yourself out if you see something unfamiliar on the test—with a little finagling, you will be able to change it into a familiar form
- Don't be overwhelmed by the number of practice problems you think you have to do—it is far more important to understand concepts. It is much more important to understand concepts than to memorize ways to do problems.
- If you do want good practice problems, old worksheets are a good place to look.
- Get a good night's sleep.
- Eat lots of candy, or find some other way to relax.

Applications of Derivative concepts

$$y = \ln\left(\tan^{-1}(x^2\cos x)\right)$$

$$y' = \frac{1}{\tan^{-1}(x^2\cos x)}\,\frac{d}{dx}\left[\tan^{-1}(x^2\cos x)\right]$$

$$= \frac{1}{\tan^{-1}(x^2\cos x)}\left(\frac{1}{1+(x^2\cos x)^2}\right)\frac{d}{dx}\left[x^2\cos x\right]$$

$$= \frac{1}{\tan^{-1}(x^2\cos x)}\left(\frac{1}{1+(x^2\cos x)^2}\right)\left(2x\cos x - x^2\sin x\right)$$

Logarithmic Differentiation

$$y = \frac{ex^2\sin x}{\sqrt{x^2+6x+8}}$$ ← Good to use log. diff b/c we have a rational expression w/ lots of complicated things!

$$\ln y = \ln\left(\frac{ex^2\sin x}{(x^2+6x+8)^{\frac{1}{2}}}\right)$$

$$\ln y = \ln ex^2\sin x - \ln(x^2+6x+8)^{\frac{1}{2}}$$

$$\ln y = \ln e + 2\ln x + \ln\sin x - \tfrac{1}{2}\ln(x^2+6x+8)$$

$$\frac{1}{y}y' = \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2}\left(\frac{1}{x^2+6x+8}\right)(2x+6)$$

$$\frac{1}{y}y' = \frac{2}{x} + \cot x - \frac{x+3}{x^2+6x+8}$$

$$y' = \underbrace{\left(\frac{ex^2\sin x}{\sqrt{x^2+6x+8}}\right)}_{y}\left(\frac{2}{x} + \cot x - \frac{x+3}{x^2+6x+8}\right)$$

Implicit Differentiation

$$\cos(xy) = xy^2 + 2x$$

$$-\sin(xy)\left[y + x\frac{dy}{dx}\right] = x\,2y\frac{dy}{dx} + y^2 + 2$$ Solve for $\frac{dy}{dx}$!

$$-y\sin(xy) - y^2 - 2 = 2xy\frac{dy}{dx} + x\sin(xy)\frac{dy}{dx} = \left(2xy + x\sin(xy)\right)\frac{dy}{dx}$$

$$\frac{-y\sin(xy) - y^2 - 2}{2xy + x\sin(xy)} = \frac{dy}{dx}$$

②

$x^2 + x^2y + y^2 = 0$ Find equation of tangent line when $x=2$

$(y - y_0 = m(x - x_0))$

$2x + x^2\frac{dy}{dx} + 2xy + 2y\frac{dy}{dx} = 0$

$(x^2 + 2y)\frac{dy}{dx} = -2x - 2xy$

$\frac{dy}{dx} = \frac{-2x(1+y)}{x^2 + 2y}$

Need to find y:

$4 + 4y + y^2 = 0$

$(2+y)(2+y) = 0$

$y = -2$

So $\frac{dy}{dx}$ when $x=2$ is:

$\frac{-2(2)(1-2)}{2^2 - 4}$ ← undefined

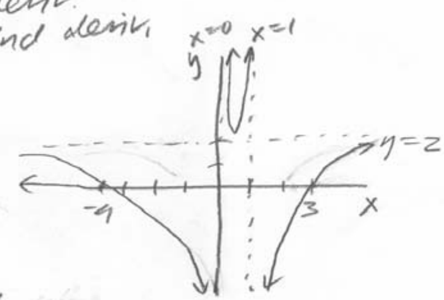
we know we have a vertical tangent line → eq. of tan. line is $x=2$

Graphs

Important things to know about graphs

1) root(s), y-intercept
2) vertical/horizontal asymptotes ← $\lim_{x \to \pm\infty}$
3) critical points - 1st deriv.
4) Inflection points - 2nd deriv.

EX: $\frac{2(x+4)(x-3)}{x(x-1)}$



V.A. when $x=0,1$

roots $x=-4, 3$

horizontal asymptote

$\lim_{x \to \pm\infty} \frac{2(x+4)(x-3)}{x(x-1)} = 2$

(Graphs cont'd)

3

Ex: $g(x) = x^3 + 3x^2 + 3x - 21$

find max, min:

$$g'(x) = 3x^2 + 6x + 3 = 0$$
$$x^2 + 2x + 1 = 0$$
$$(x+1)^2 = 0$$

critical pts: $x = -1$

Increasing test:

$+ \quad + $

-1

slope of tangent line is 0, point of inflection

max: change from inc. to dec. ($g'(x)$ $(+) \to (-)$)

min: change from dec. to inc. ($g'(x)$ $(-) \to (+)$)

$$g''(x) = 6x + 6 = 0$$
$$x = -1$$

Concavity test:

$- \quad +$

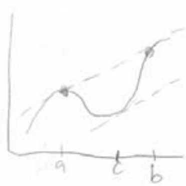
-1

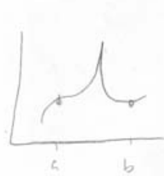
concave down

concave up

Mean Value Theorem

f is differentiable on $[a, b]$ $\longrightarrow$ there exists a value c between a and b such that $f'(c) = \frac{f(b) - f(a)}{b - a}$





Not differentiable

Does not satisfy the hypothesis

$f'(x) \le 5$

$a = 2 \quad f(a) = 16$

$b = 5 \quad f(b) = \,?$ ← want the largest value

Take largest possible slope

$$5 = \frac{f(b) - 16}{3} \qquad 15 = f(b) - 16$$
$$31 = f(b)$$

Extreme Value Theorem

If you have a continuous function on a closed interval $[a,b]$, then it attains a maximum and minimum value

Fermat's Thm - slope of tangent line of relative extrema (max, min) is 0!

L'Hospital's Rule $\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$ ← only for $\frac{0}{0}$, $\frac{\infty}{\infty}$

$\left.\begin{array}{l} \lim_{x\to 0} \frac{\sin x}{x} = 1 \\ \lim_{x\to 0} \frac{\cos x - 1}{x} = 0 \end{array}\right\}$ really important! (esp. first one)

Indeterminate Forms

$\underbrace{\frac{0}{0} \quad \frac{\infty}{\infty}}_{\text{only ones w/ which we can apply rule}} \qquad \infty^0 \quad 1^\infty \quad 0\cdot\infty$

Non-Indeterminate Forms (careful!)

$\frac{0}{\infty} \quad \frac{\infty}{0} \quad 0^\infty \quad \infty\cdot\infty \quad +\infty+\infty \quad \underbrace{\infty-(-\infty)}_{=+\infty+\infty}$

$$\lim_{x\to\infty} e^{x} x = \lim_{x\to\infty} \frac{e^{x}}{x^{-1}} = \lim_{x\to\infty} \frac{x}{e^{-x}}$$

$\lim_{x\to 0^+} \sin x \ln x$ answer $= 0$

$\lim_{x\to 0^+} (\tan 2x)^x$ answer $= e^0$

← exercises

$$\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx}$$

Call $y = \left(1+\frac{a}{x}\right)^{bx}$

$\ln y = bx \ln\left(1+\frac{a}{x}\right)$

Can do this since $\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y}$

$$\lim_{x\to\infty} bx \ln\left(1+\tfrac{a}{x}\right) = \lim_{x\to\infty} \frac{b\ln\left(1+\frac{a}{x}\right)}{x^{-1}} \quad \leftarrow \frac{0}{0}$$

Use L'Hopital's: $$\lim_{x\to\infty} \frac{b\left(\frac{1}{1+\frac{a}{x}}\right)\left(\frac{-a}{x^2}\right)}{-x^{-2}} = \lim_{x\to\infty} \frac{ab}{1-\frac{a}{x}} = ab$$

So $\lim_{x\to\infty} y = e^{ab}$

Linearization

key idea: near a point, a function can be approximated by the tangent line

$x =$ unknow
$a =$ known point
$f(x) = f(a) + f'(a)(x-a)$

Ex: Approximate $(2.001)^5$

$f(x) = x^5 \qquad f'(x) = 5x^4$

$f(x) = f(a) + f'(a)(x-a)$

$f(2.001) = 2^5 + 80(2.001-2) = 32 + 8(.001) = \boxed{32.08}$

our approximation

relative error = $\dfrac{\text{absolute error}}{\text{actual value}}$

$$V = \tfrac{4}{3}\pi r^3$$
$$\frac{dV}{dr} = 4\pi r^2$$
$$dV = 4\pi r^2\, dr$$
$$\frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3}$$

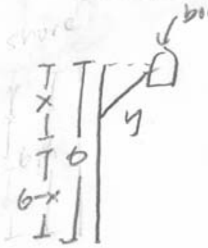
Related Rates

(see first sheet)

Optimization

1) Draw a picture
2) Figure out known/unknown, what is a variable (changing) and what's constant
3) Make an equation
4) Find the key relationship
5) substitute to make a function on one variable
6) Optimize

Example: In a boat 2 mi from shore, 6 mi downshore is an outhouse. Row 2 mph, run 6 mph. What is fastest way to outhouse?



$$T = \frac{d}{v}$$

$$T = T_{water} + T_{land}$$

$$T = \frac{y}{2} + \frac{6-x}{6} \leftarrow \text{Now need function of one variable}$$

But we know: $2^2 + x^2 = y^2$

$$y = \sqrt{x^2+4}$$

$$T(x) = \frac{\sqrt{x^2+4}}{2} + \frac{6-x}{6}$$

$$T'(x) = \tfrac{1}{2}\cdot\tfrac{1}{2}(x^2+4)^{-\frac{1}{2}}(2x) - \tfrac{1}{6}$$

$$0 = \frac{2x}{4\sqrt{x^2+4}} - \frac{1}{6}$$

$$\frac{1}{6} = \frac{x}{2\sqrt{x^2+4}} \qquad \boxed{x = \sqrt{\tfrac{1}{2}}}$$

etc. etc. etc.

(Remember to verify min. w/ D

(SAVE PAPER!)