Name: ______________________ ID#: ______________

Solutions to Final Exam

Math 1a
Introduction to Calculus

21 January 2005

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

Please check your section:

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Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.

—*Handbook for Students*

1. (18 Points) *Compute the following limits, with justification.*

(i) $\lim_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x^2+7}$

Solution. We have

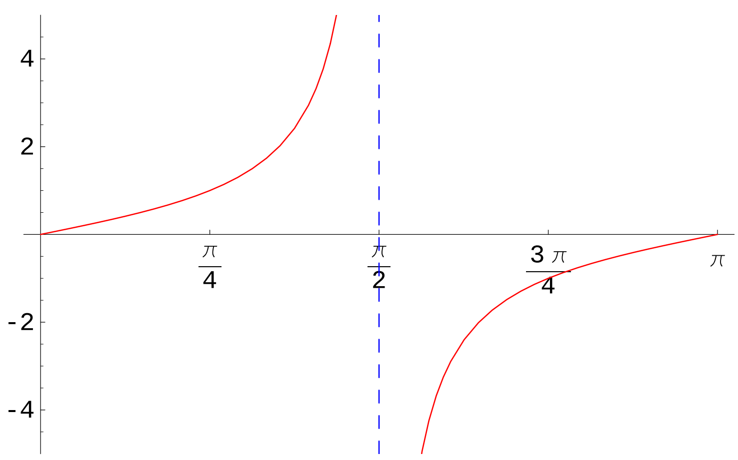
$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x^2+7} &= \lim_{x\to\infty} \frac{x\sqrt{1+1/x^2}}{x^2(1+7/x^2)} \\ &= \lim_{x\to\infty} \frac{1}{x} \cdot \lim_{x\to\infty} \frac{\sqrt{1+1/x^2}}{1+7/x^2} \\ &= 0 \cdot \frac{\sqrt{1+0}}{1+0} = 0.\end{aligned}$$

□

(ii) $\lim_{x\to\pi/2^+} \tan(x)$

Solution. As $x \to \frac{\pi}{2}$, $\cos x$ gets small, while $\sin x$ is close to 1. So values of $\tan x$ increase without bound. We have

$$\lim_{x\to\pi/2^-} \tan(x) = \infty \qquad \lim_{x\to\pi/2^+} \tan(x) = -\infty$$



□

(iii) $\lim_{x\to 1} \frac{x-1}{e^x-1}$

Solution. By the Direction Substitution Property,

$$\lim_{x\to 1} \frac{x-1}{e^x-1} = \frac{1-1}{e-1} = 0.$$

The hypotheses of L'Hôpital's Rule are not satisfied, so using it will produce a wrong answer. □

(iv) $\lim_{x\to 0} \frac{2\sec(x)-2-x^2}{x^2}$

Solution. We have

$$\begin{aligned}\lim_{x\to 0} \frac{2\sec(x)-2-x^2}{x^2} &\overset{\text{H}}{=} \lim_{x\to 0} \frac{2\sec x\tan x - 2x}{2x} \\ &\overset{\text{H}}{=} \lim_{x\to 0} \frac{2(\sec x\tan^2 + 2\sec^3) - 2}{2} \\ &= \frac{2(1\cdot 0^2 + 1^3) - 2}{2} = 0.\end{aligned}$$

□

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2. (8 Points) *Let f be the function $f(x) = \sin(x) - \frac{x}{3}$.*

(a) *Use the Intermediate Value Theorem to show that there exists a point c in $\left(\frac{\pi}{2}, \pi\right)$ such that $f(c) = 0$.*

Solution. Since $f\left(\frac{\pi}{2}\right) = 1 - \frac{\pi}{6} < 0$ and $f(\pi) = 0 - \frac{\pi}{3} > 0$, there must be a point in between the two such where $f = 0$. □

(b) *Suppose that there was another point d in the same interval such that $f(d) = 0$. What fact would the Mean Value Theorem allow you to conclude? Why is this "fact"* impossible*?*

Solution. Let's assume d existed which was different from c. Then by the Mean Value Theorem there would be a point p between c and d such that

$$f'(p) = \frac{f(c) - f(d)}{c - d} = 0.$$

But for all x,

$$f'(x) = \cos x - \frac{1}{3},$$

and for x in $\left(\frac{\pi}{2}, \pi\right)$ this is negative. We have reached a contradiction, so d cannot exist. Hence c is the unique solution in the interval to the equation $f(x) = 0$.

□

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3. (20 Points) *Find the following derivatives.*

(i) $\dfrac{d}{dx}(3x^2 + 4x + 6)$

Solution. By the power rule,

$$\frac{d}{dx}(3x^2 + 4x + 6) = 6x + 4$$

□

(ii) $\dfrac{d}{dx}\dfrac{2^x}{1 + \ln x}$

Solution. By the quotient rule,

$$\frac{d}{dx}\frac{2^x}{1 + \ln x} = \frac{(1 + \ln x)(\ln 2)2^x - 2^x(1/x)}{(1 + \ln x)^2}.$$

□

(iii) $\frac{d}{dx}\cos\left(x^{1/3}\right)$

Solution. By the chain rule, we have

$$\frac{d}{dx}\cos\left(x^{1/3}\right) = -\sin\left(x^{1/3}\right)\cdot\frac{1}{3}x^{-2/3}.$$

□

(iv) $\frac{d}{dx}x^{\sqrt{x}}$

Solution. Let $y = x^{\sqrt{x}}$, so

$$\begin{aligned} \ln y &= \sqrt{x}\ln x \\ \frac{1}{y}\frac{dy}{dx} &= \frac{1}{2\sqrt{x}}\cdot\ln x + \sqrt{x}\cdot\frac{1}{x} \\ \implies \frac{dy}{dx} &= \left(\frac{1}{2\sqrt{x}}\cdot\ln x + \frac{1}{\sqrt{x}}\right)x^{\sqrt{x}} \end{aligned}$$

□

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4. (8 Points)

(a) *Write down the function which is the linear approximation to the square root function at* $a = \frac{9}{4}$.

Solution. Let $f(x) = \sqrt{x}$. Then the linear approximatino to f at a is

$$\begin{aligned} L(x) &= f\left(\tfrac{9}{4}\right) + f'\left(\tfrac{9}{4}\right)\left(x - \tfrac{9}{4}\right) \\ &= \tfrac{3}{2} + \tfrac{1}{3}\left(x - \tfrac{9}{4}\right) \end{aligned}$$

□

(b) *Use this function to approximate* $\sqrt{2}$.

Solution. We only need to find L at 2:

$$L(2) = \frac{3}{2} + \frac{1}{3}\left(x - \frac{9}{4}\right) = \frac{3}{2} + \frac{1}{3}\left(-\frac{1}{4}\right) = \frac{3}{2} - \frac{1}{12} = \frac{17}{12}.$$

Now $\frac{17}{12} \approx 1.41667$, which agrees with $\sqrt{2}$ to two decimal places. □

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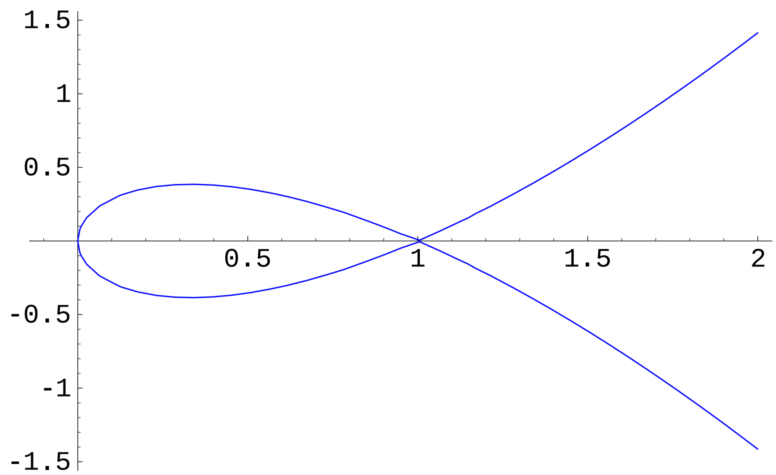
5. (15 Points) *The relation $y^2 = x(x-1)^2$ defines a curve in the plane.*



(i) Solve for y in terms of x, and use it to find $\frac{dy}{dx}$ at the point $\left(\frac{1}{4}, -\frac{3}{8}\right)$.

Solution. We have

$$y = \pm\sqrt{x(x-1)^2} = \sqrt{x^3 - 2x^2 + x}$$
$$\implies \frac{dy}{dx} = \pm\frac{3x^2 - 4x + 1}{2\sqrt{x^3 - 2x^2 + x}},$$

which at $x = \frac{1}{4}$ is $\pm\frac{1}{4}$. Which one is it? Since we are looking at the point in the fourth quadrant, where $y < 0$, we need the negative choice. □

(ii) Find $\dfrac{dy}{dx}$ *implicitly in terms of* y *and* x*. What is its value at* $\left(\frac{1}{4}, -\frac{3}{8}\right)$*?*

Solution. We have

Solution.

$$2y\frac{dy}{dx} = 3x^2 - 4x + 1$$
$$\implies \frac{dy}{dx} = \frac{3x^2 - 4x + 1}{2y},$$

which at $\left(\frac{1}{4}, -\frac{3}{8}\right)$ is $-\frac{1}{4}$. Using implicit differentation allowed us to not have to worry about choices. □

□

(iii) A parametrization of the curve is given by

$$x(t) = t^2 \qquad\qquad y(t) = t(t^2 - 1).$$

Find $\dfrac{dy}{dx}$ *in terms of* t*. What is its value at* $t = \frac{1}{2}$*? (This corresponds to the point* $\left(\frac{1}{4}, -\frac{3}{8}\right)$*).*

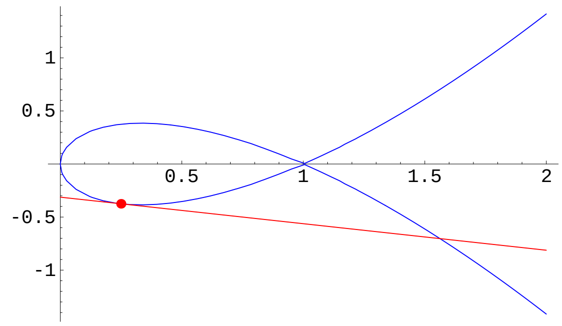
Solution. We have

$$\frac{dx}{dt} = 2t \qquad\qquad \frac{dy}{dt} = 3t^2 - 1$$

So

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 1}{2t},$$

which at $t = \frac{1}{2}$ is $-\frac{1}{4}$. □



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6. (18 Points) ***The dreaded graphing problem.*** *Let*

$$f(x) = \frac{1}{x+1} - \frac{1}{(x+1)^2}.$$

(a) Find all horizontal and vertical asymptotes of f.

Solution. A vertical asymptote can happen where denominators go to zero, and in this case we have to check $x = -1$. Notice that

$$f(x) = \frac{x+1}{(x+1)^2} - \frac{1}{(x+1)^2} = \frac{x}{(x+1)^2},$$

so if x is near -1, then x is negative, and $(x+1)^2$ is very small and positive. So

$$\lim_{x \to -1} f(x) = -\infty,$$

confirming that $x = -1$ is a vertical asymptote.

For horizontal asymptotes, we need to find

$$\lim_{x \to \infty} \frac{x}{(x+1)^2} = 0,$$

and the same will be true for the limit as $x \to -\infty$. Hence $y = 0$ is a vertical asymptote, and there are no others. □

(b) The derivative of f is

$$f'(x) = -\frac{1}{(x+1)^2} + \frac{2}{(x+1)^3}.$$

Find the intervals of increase or decrease.

Solution. By algebra we have

$$f'(x) = \frac{1-x}{(x+1)^3}.$$

The points of interest are $x = 1$ (a critical point) and $x = -1$ (f' is not defined there). We can make a sign chart as follows:

	$x < -1$	$-1 < x < 1$	$x > 1$
$1 - x$	$+$	$+$	$-$
$(x+1)^3$	$-$	$+$	$+$
$f'(x)$	$-$	$+$	$-$
f	$\searrow$	$\nearrow$	$\searrow$

So f is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, \infty)$. $\square$

(c) Find any local maxima or minima.

Solution. We have a local maximum at $\left(1, \frac{1}{4}\right)$. $\square$

(d) The second derivative of f is

$$f''(x) = \frac{2}{(x+1)^3} - \frac{6}{(x+1)^4}$$

Find the intervals of concavity.

Solution. Simplifying, we have

$$f''(x) = \frac{2(x-2)}{(x+1)^2},$$

so the points of interest are $x = 2$ and $x = -1$. The sign chart looks like this:

	$x < -1$	$-1 < x < 2$	$x > 2$
2	$+$	$+$	$+$
$x - 2$	$-$	$-$	$+$
$(x+1)^4$	$+$	$+$	$+$
$f''(x)$	$-$	$-$	$+$
f	$\frown$	$\frown$	$\smile$

So f is concave up on $(2, \infty)$ and concave down on $(-\infty, -1)$ and $(-1, 2)$. One may *not* say that f is concave down on $(-\infty, 2)$ because f is not defined at $x = -1$. □

(e) Find any inflection point(s).

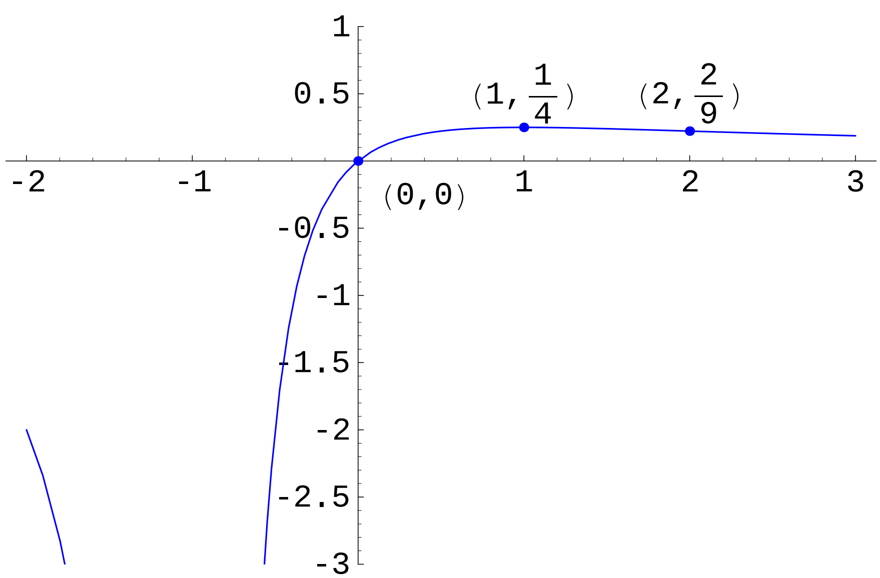
Solution. Apparently $\left(2, \frac{2}{9}\right)$ is the only inflection point. □

(f) Sketch the graph of f. Label all the significant points you have found previously.

Solution.

- On $(-\infty, -1)$, f is negative, decreasing, and concave down, approaching a limit of $-\infty$.
- At $x = -1$ there is a vertical asymptote.
- On $(-1, 1)$, f is increasing and concave down, crossing the x-axis at $x = 0$.
- At $x = 1$, f reaches a local maximum.
- On $(1, 2)$, f is positive, decreasing, and concave down.
- At $x = 2$, f has an inflection point.
- On $(2, \infty)$, f is decreasing, concave up, and tends towards a limit of 0.

Putting this all together, we have the following graph for f:



□

(g) Find the global minimum and maximum, if they exist.

Solution. Because $\lim_{x \to -1} f(x) = -\infty$, there is no global minimum. Apparently $\left(1, \frac{1}{4}\right)$ is the global maximum. □

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Figure 1: Stacy

7. (18 Points) *Apex Corporation is planning to sell sponges on television. The sponges cost \$2 per package. Stacy believes that if they set the selling price at \$20 per package, they will sell* 1000 *packages, and for every dollar they increase the price, the quantity they will sell will decrease by* 50 *packages.*

We will find the price at which profit will be maximized.

(a) Assuming Stacy's assumptions about the market are true, show that the demand curve (price in terms of quantity sold) is given by

$$p(x) = 40 - \tfrac{1}{50}x.$$

How many packages will be sold if the price is set at \$27?

Solution. We can use the point-slope formula for the line:

$$p - 20 = \frac{-1}{50}(x - 1000),$$

which simplifies to what was given. If $p = 27$, we have $x = 650$. □

(b) Now show that the profit (this is revenue minus costs, remember) is given by

$$K(x) = -\tfrac{1}{50}x(x - 1900)$$

Solution. If x units are sold at price $p(x)$, the costs are $2x$ and the revenue is $R(x) = x \cdot p(x)$, and the profit is

$$K(x) = x\left(40 - \tfrac{1}{50}x\right) - 2x,$$

which simplifies to what was given. □

(c) What price maximizes profit? Make sure you show it's maximal and not minimal!

Solution. We will use the closed interval test on the interval $[0, 2000]$, which are the limits of the feasibility of the demand curve (if $x > 2000$, the price predicted is negative). We have $K(0) = \$0$ and $K(2000) = -\$4000$. To find the critical points, we find $K'(x)$:

$$K(x) = 38x - \frac{x^2}{50}$$
$$K'(x) = 38 - \frac{x}{25},$$

which is zero when $x = 38 \cdot 25 = 950$. $K(950) = \$18,050$, so this is the maximum value on the interval.

To find what *price* maximizes profit, we go back to the demand curve. Apparently $p(950) = \$21$. □

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8. (8 Points) *Ferdbert Freshman is studying for his Math 1a Final. He starts studying at midnight and does problems at the rate of*

$$r(t) = \frac{60}{\pi(t^2+1)}$$

problems per hour, where t *is measured in hours after midnight. How many problems has he done by 1:00* AM*?*

Solution. The task is to compute

$$\int_0^1 \frac{60}{\pi(t^2+1)} dt.$$

Since $\arctan t$ is an antiderivative of $\frac{1}{t^2+1}$, it follows from the Fundamental Theorem of Calculus that

$$\int_0^1 \frac{60}{\pi(t^2+1)} dt = \frac{60}{\pi} \arctan t \Big|_0^1. \tag{1}$$

Substituting $\pi/4 = \arctan 1$ and $0 = \arctan 0$ in (1) yields 15; i.e., Ferdbert has solved 15 problems by 1:00 AM. □

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9. (8 Points) *Evaluate the following definite integrals.*

(i) $\int_1^4 (2x + x^2)\, dx$

Solution. We have

$$\int_1^4 (2x + x^2) dx = (x^2 + \frac{1}{3}x^3)\Big|_1^4 = 36.$$

□

(ii) $\int_1^{e^{17}} \frac{1}{x}\, dx$

Solution. We have

$$\int_1^{e^{17}} \frac{1}{x} dx = \ln x|_1^{e^{17}} = \ln e^{17} - \ln 1 = 17.$$

□

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10. (15 Points) *Compute the following integrals. For definite integrals, your answer should be a number. For indefinite integrals, your answer should be the most general antiderivative as a function of x.*

(i) $\int \left(3x\sqrt{x^2+1}\right)\,dx$

Solution. Let $u = x^2 + 1$, so $du = 2x\,dx$. Then

$$\begin{aligned}\int \left(3x\sqrt{x^2+1}\right)\,dx &= \frac{3}{2}\int \sqrt{u}\,du \\ &= \frac{3}{2}\cdot\frac{2}{3}u^{3/2} + C = \sqrt{x^2+1} + C\end{aligned}$$

□

(ii) $\int \frac{\ln x}{x}\,dx$

Solution. Let $u = \ln x$, so that $du = \frac{dx}{x}$. Hence

$$\begin{aligned}\int \frac{\ln x}{x}\,dx &= \int u\,du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}(\ln x)^2 + C.\end{aligned}$$

□

(iii) $\int_0^1 \frac{e^{4x}}{1+e^{4x}}\,dx$

Solution. Let $u = 1 + e^{4x}$, so that $du = 4e^{4x}\,dx$. Then

$$\begin{aligned}\int_0^1 \frac{e^{4x}}{1+e^{4x}}\,dx &= \frac{1}{4}\int_2^{e^4+1} \frac{du}{u} \\ &= \frac{1}{4}\ln u\Big|_2^{e^4+1} \\ &= \frac{1}{4}\left(\ln(e^4+1) - \ln 2\right) = \frac{1}{4}\ln\left(\frac{1}{2}(e^4+1)\right).\end{aligned}$$

□

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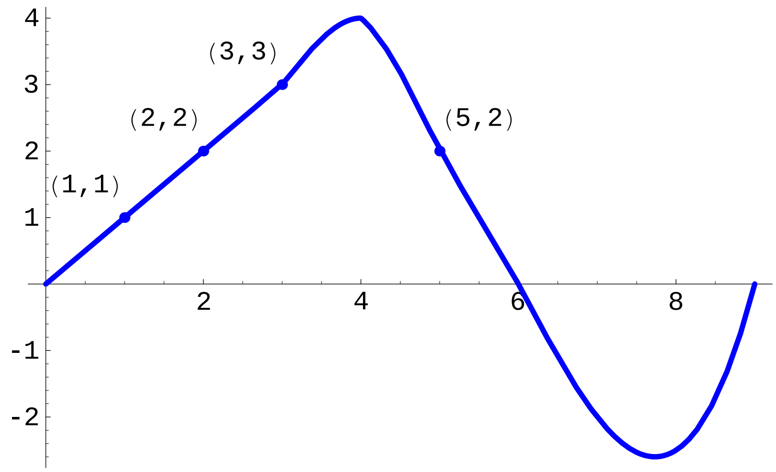
11. (14 Points) *Suppose that f is the differentiable function shown in the graph below*



(The function is a straight line from $(0,0)$ to $(3,3)$, and is differentiable at $x = 4$, even though the graph looks a little pointy.) Suppose the the position at time t seconds of a particle moving along a coordinate axis is

$$s(t) = \int_0^t f(x)\,dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.

(i) *What is the particle's velocity at time $t = 5$?*

Solution. Recall that by the Fundamental Theorem of Calculus we have

$$s'(t) = f(t).$$

So $s'(5) = f(5) = 2$. □

(ii) *Is the acceleration of the particle at time $t = 5$ positive or negative?*

Solution. We have $s''(5) = f'(5)$, which looks negative from the graph. □

(iii) *What is the particle's position at time $t = 3$?*

Solution. Since on $[0, 3]$, $f(x) = x$, we have

$$s(3) = \int_0^3 x\,dx = \frac{9}{2}.$$

□

(iv) *At what time during the first 9 seconds does s have its largest value?*

Solution. The critical points of s are the zeros of $s' = f$. By looking at the graph, we see that f is positive from $t = 0$ to $t = 6$, then negative from $t = 6$ to $t = 9$. Therefore s is increasing on $[0, 6]$, then decreasing on $[6, 9]$. So its largest value is at $t = 6$. □

(v) *Approximately when is the acceleration zero?*

Solution. $s'' = 0$ when $f' = 0$, which happens at $t = 4$ and $t = 7.5$ (approximately) □

(vi) *When is the particle moving toward the origin? Away from the origin?*

Solution. The particle is moving away from the origin when $s > 0$ and $s' > 0$. Since $s(0) = 0$ and $s' > 0$ on $(0, 6)$, we know the particle is moving away from the origin then. After $t = 6$, $s' < 0$, so the particle is moving toward the origin. □

(vii) *On which side (positive or negative) of the origin does the particle lie at time $t = 9$?*

Solution. We have

$$s(9) = \int_0^6 f(x)\,dx + \int_6^9 f(x)\,dx,$$

where the left integral is positive and the right integral is negative. In order to decide whether $s(9)$ is positive or negative, we need to decide if the first area is more positive than the second area is negative. This appears to be the case, so $s(9)$ is positive. □

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