Name: ______________________ ID#: ______________

Solutions to Final Exam

Math 1a
Introduction to Calculus

21 May 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

> *Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.*
>
> —*Handbook for Students*

1. (15 Points) *Find the following limits.*

(i) $\lim_{x\to\infty} \frac{1+x}{1-x}$

Solution. Since this limit is of the form $\frac{\infty}{\infty}$, we can use L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{1+x}{1-x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1}{-1} = -1.$$

□

(ii) $\lim_{x\to 2} \frac{x^2-4}{x^2+4}$

Solution. This limit is not of an indeterminate form, since the numerator goes to 0 whilst the denominator goes to 8. Hence by the ordinary limit laws, the limit is 0.

Using L'Hôpital's Rule gives an answer of 1, which is incorrect. □

(iii) $\lim_{x\to 0^+} (\cos x)^{1/x}$

Solution. This limit is of the form 1^∞, which is indeterminate. Hence we can use L'Hôpital's Rule on its logarithm. If $\lim_{x\to 0^+} (\cos x)^{1/x} = L$, we have

$$\begin{aligned}
\ln L &= \ln \lim_{x\to 0^+} (\cos x)^{1/x} \\
&= \lim_{x\to 0^+} \ln(\cos x)^{1/x} \\
&= \lim_{x\to 0^+} \frac{\ln(\cos x)}{x} \\
&\overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{1} \\
&= -\lim_{x\to 0^+} \tan x = 0.
\end{aligned}$$

Therefore $L = e^0 = 1$. □

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2. (20 Points) *Find the derivatives of the following functions.*

(i) $f(x) = x^3$

Solution. By the power rule, we have

$$f'(x) = 3x^2.$$

□

(ii) $f(x) = 3^x$.

Solution. By the exponential rule, we have

$$f'(x) = 3^x(\ln 3).$$

□

(iii) $f(x) = \sin(x^2)$

Solution. By the chain rule, we have

$$f'(x) = \cos(x^2)(2x).$$

□

(iv) $f(x) = (\sin x)^2$

Solution. Again by the chain rule, but with the composition of functions in the opposite order, we have

$$f'(x) = 2(\sin x)(\cos x).$$

□

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3. (15 Points) *A fish tank is to be designed with square base and rectangular sides (and no top). The material for the base costs five times as much as the glass for the sides. What are the dimensions of the tank which has volume* $20ft^3$ *and costs the least to build?*

Solution. The volume of the tank is the function $V = x^2h$, where x is the length (and width) of the base and h is the height. Since V is fixed to be 20ft^3, we have

$$h = \frac{V}{x^2} = \frac{20}{x^2}.$$

If c is the cost of a square foot of glass, then $5c$ is the cost of a square foot of the base material. This makes the total cost of the tank

$$C = 5x^2(5c) + 4xhc = \left(5x^2 + \frac{80}{x}\right)c$$

dollars. We have

$$\frac{dC}{dx} = \left(10x - \frac{80}{x^2}\right)c,$$

which is zero when

$$\begin{aligned} 10x &= \frac{80}{x^2} \\ \implies x^3 &= 8 \\ \implies x &= 2\text{ft}. \end{aligned}$$

Thus the length and width of the box are 2 feet and the height is 5 feet. □

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4. (15 Points) 1 *is a pretty lousy approximation to* $\sqrt[3]{2}$. *Using Newton's method, find a rational number (whole number or fraction) which is closer, and another one which is still closer. Simplify your answer to a reasonable form.*

Solution. In order to use Newton's method to find $\sqrt[3]{2}$, we need a function which is zero at $\sqrt[3]{2}$. One choice would be $f(x) = x - \sqrt[3]{2}$, but that involves the number we're trying to find itself (also, $\sqrt[3]{2}$ isn't rational). Instead, try $f(x) = x^3 - 2$, so that $f(\sqrt[3]{2}) = 0$. Iterating Newton's method with points close to $\sqrt[3]{2}$ should bring us even closer. We create the Newton function

$$N_f(x) = x - \frac{x^3 - 2}{3x^2} = \frac{2}{3}\left(x + \frac{1}{x^2}\right)$$

and iterate it. A better approximation than 1 is

$$N_f(1) = \frac{2}{3}\left(1 + \frac{1}{1^2}\right) = \frac{4}{3},$$

and one which is still better is

$$N_f(2) = \frac{2}{3}\left(\frac{4}{3} + \frac{9}{16}\right) = \frac{91}{72}.$$

□

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5. (20 Points) *Jake and Elwood are organizing a concert to benefit the orphanage in which they grew up. They have determined through market research that if they set the price of tickets to be \$1, they will sell 7500 tickets, and if they set the price of tickets to be \$3, they will sell 2500 tickets.*

(a) *Assuming that the demand function (number of tickets sold in terms of price charged) $x(p)$ is linear, show that*

$$x(p) = 10,000 - 2500p.$$

Solution. We have two points on the demand line: $(1, 7500)$ and $(3, 2500)$. The slope of this line is

$$m = \frac{5000}{2} = 2500.$$

Thus, using point-slope form (here x is the "y-value" and p is the "x-value")

$$x - 2500 = 2500(p - 3),$$

which simplifies to what we want. □

(b) *Recalling that*

$$\textit{revenue} = \textit{quantity sold} \times \textit{price charged},$$

what ticket price will maximize their revenue?

Solution. The equation for revenue comes

$$R = (10,000 - 2500p)p = 10,000p - 2500p^2.$$

Revenue is optimized when $R'(p) = 0$, or

$$0 = 10,000 - 5000p \implies p = 2.$$

So Jake and Elwood should set the price at \$2; they will sell 5000 tickets and gross \$10,000.

This is from the great Chicago classic *The Blues Brothers.* □

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6. (15 Points) *Show that*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2+i^2} = \frac{\pi}{4}.$$

Hint. $\dfrac{n}{n^2+i^2} = \dfrac{1}{1+\left(\frac{i}{n}\right)^2}\dfrac{1}{n}.$

Solution. Using the hint, we rewrite the limit as

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{n}\right)^2}\frac{1}{n},$$

which looks kind of like a limit of Riemann sums. For each n, $\displaystyle\sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{n}\right)^2}\frac{1}{n}$ is the Riemann sum for the function $f(x) = \dfrac{1}{1+x^2}$ using $x_i^* = x_i$. So the limit is the definite integral

$$\int_0^1 \frac{1}{1+x^2}\, dx = \arctan x\Big|_0^1 = \frac{\pi}{4}.$$

□

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7. (20 Points) *Compute the following integrals. For definite integrals, your answer should be a number. For indefinite integrals, your answer should be the most general antiderivative as a function of x.*

(i) $\int_1^2 (x^3+4)\,dx$

Solution. This is a straightforward application of the power rule.

$$\begin{aligned}\int_1^2 (x^3+4)\,dx &= \frac{x^4}{4}+4x+C\Big|_1^2 \\ &= \left[\frac{2^4}{4}+4(2)\right]-\left[\frac{1^4}{4}+4(1)\right] \\ &= \frac{31}{4}.\end{aligned}$$

□

$\int xe^{-x^2}\,dx$

Solution. Let $u=x^2$, so $du=2x\,dx$, and $x\,dx=\frac{1}{2}du$. Thus

$$\begin{aligned}\int xe^{-x^2}\,dx &= \frac{1}{2}\int e^{-u}\,du \\ &= -\frac{1}{2}e^{-u}+C=-\frac{1}{2}e^{-x^2}+C.\end{aligned}$$

□

(ii) $\int_1^2 \frac{(\ln x)^6}{x}\,dx$

Solution. Let $u=\ln x$, so $du=\frac{dx}{x}$. Thus

$$\begin{aligned}\int_1^2 \frac{(\ln x)^6}{x}\,dx &= \int_0^{\ln 2} u^6\,du \\ &= \frac{u^7}{7}\Big|_0^{\ln 2}=\frac{(\ln 2)^7}{7}.\end{aligned}$$

□

$\int \sin^3 x\,dx$

Hint. $\sin^2 x = 1-\cos^2 x$.

Solution. Using the hint, we have

$$\begin{aligned}\int \sin^3 x\, dx &= \int \sin x(1-\cos^2 x)\, dx \\ &= \int \sin x\, dx - \int \sin x \cos^2 x\, dx.\end{aligned}$$

The first integrand can be integrated directly, while the second one requires a $u = \cos x$, $du = -\sin x\, dx$ substitution. We arrive at

$$\int \sin^3 x\, dx = -\cos x + \frac{1}{3}\cos^3 x + C.$$

□

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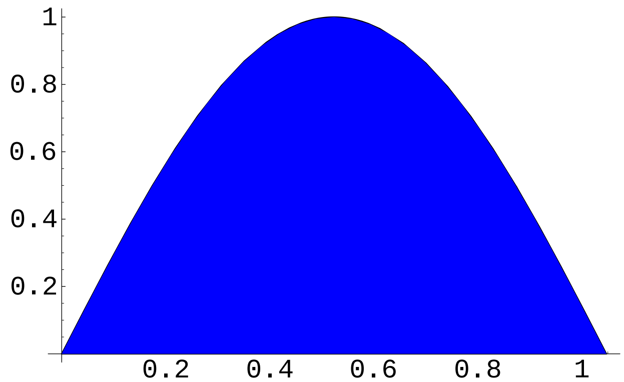
8. (15 Points)

(a) (5 points) Draw the region between the graph of $y = \sin 3x$, the x-axis, and the vertical lines $x = 0$ and $x = \pi/3$.

Solution.



□

(b) (10 points) Find its area.

Solution. We have

$$A = \int_0^{\pi/3} \sin 3x \, dx$$

$$= -\frac{1}{3} \cos 3x \Big|_0^{\pi/3}$$

$$= -\frac{1}{3} \left((-1) - 1 \right) = \frac{2}{3}.$$

□

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9. (15 Points) *Let k be a constant and*

$$F(y) = \int_0^y \frac{dt}{\sqrt{1-k^2\sin^2 t}}.$$

(a) Find $F'(y)$. (It will have a k in it, but not a t.)

Solution. It is impossible to compute the antiderivative in elementary terms. The way to find the derivative of this area function is to use the Fundamental Theorem of Calculus:

$$F'(y) = \frac{1}{\sqrt{1-k^2\sin^2 y}}.$$

□

Define a new function $a(x) = F^{-1}(x)$, called the amplitude *of x, so that*

$$F(a(x)) = x \qquad (*)$$

is true for all x. Also define

$$\begin{aligned} s(x) &= \sin(a(x)); \\ c(x) &= \cos(a(x)); \\ d(x) &= \sqrt{1-k^2\sin^2(a(x))} \end{aligned}$$

(these are called the Jacobi elliptic functions*).*

(b) Use (a) and $(*)$ to show

$$\frac{d}{dx}a(x) = d(x).$$

Solution. Let $y = a(x)$. It is impossible to find an explicit formula for $a(x)$. But we can find its derivative by differentiating $(*)$ implicitly. We have

$$\begin{aligned} \frac{dF}{dy}\frac{dy}{dx} &= 1 \\ \implies \frac{1}{\sqrt{1-k^2\sin^2 y}}\frac{dy}{dx} &= 1 \\ \implies \frac{dy}{dx} &= \sqrt{1-k^2\sin^2 y} = \sqrt{1-k^2\sin^2(a(x))} = d(x). \end{aligned}$$

□

(c) Show

$$\frac{d}{dx}d(x) = -k^2 s(x)c(x).$$

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Solution. This is a long chain rule application.

$$\begin{aligned}
\frac{d}{dx}d(x) &= \frac{d}{dx}\left(\sqrt{1-k^2\sin^2(a(x))}\right)^{1/2} \\
&= \frac{1}{2}\left(\sqrt{1-k^2\sin^2(a(x))}\right)^{-1/2}(-2k^2)(\sin(a(x)))(\cos(a(x)))a'(x) \\
&= -k^2\sin(a(x))\cos(a(x)) = -k^2 s(x)c(x).
\end{aligned}$$

□

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