Name: ____________________ ID#: ____________

Solutions to Midterm I

Math 1a
Introduction to Calculus

27 October 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

Please check your section:

☐	1.0	MWF10	Tatyana Chmutova	☐	4.0	TΘ10	Dawei Chen
☐	1.1	MWF10	Matthew Leingang	☐	4.1	TΘ10	Jerrel Mast
☐	2.0	MWF11	Ethan Cotterill	☐	4.2	TΘ10	Chun-Chun Wu
☐	3.0	MWF12	Matt Bainbridge	☐	5.0	TΘ11:30	Derek Bruff
				☐	5.1	TΘ11:30	Sonal Jain

Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.

—*Handbook for Students*

1. (12 Points) *Find the following limits.*

(i) $\lim_{x \to 1} \frac{x^{19} + x - 1}{\arctan(x)}$

Solution. By the limit laws, we have

$$\lim_{x \to 1} \frac{x^{19} + x - 1}{\arctan(x)} = \frac{1^{19} + 1 - 1}{\arctan(1)} == \frac{1}{\pi/4} = \frac{4}{\pi}.$$

□

(ii) $\lim_{x \to \infty} \left(\sqrt{x^4 + 19} - x^2\right)$

Solution. We can multiply and divide by the conjugate radical:

$$\begin{aligned}\lim_{x \to \infty} \left(\sqrt{x^4 + 19} - x^2\right) &= \lim_{x \to \infty} \left(\sqrt{x^4 + 19} - x^2\right) \cdot \frac{\sqrt{x^4 + 19} + x^2}{\sqrt{x^4 + 19} + x^2} \\ &= \lim_{x \to \infty} \frac{(x^4 + 19) - x^4}{\sqrt{x^4 + 19} + x^2} \\ &= \lim_{x \to \infty} \frac{19}{\sqrt{x^4 + 19} + x^2} \\ &= 0.\end{aligned}$$

□

(iii) $\lim_{x \to 19} \frac{x^3 - 19x^2 - x + 19}{x - 19}$

Solution. The numerator is zero when evaluated at 19, so is divisible by $x - 19$. In fact,

$$x^3 - 19x^2 - x + 19 = (x^2 - 1)(x - 19),$$

so

$$\begin{aligned}\lim_{x \to 19} \frac{x^3 - 19x^2 - x + 19}{x - 19} &= \frac{(x^2 - 1)(x - 19)}{x - 19} = \lim_{x \to 19} (x^2 - 1) \\ &= 361 - 1 = 360.\end{aligned}$$

□

______ / 12

2. (10 Points) *We will show that*

$$\lim_{x\to\infty} \frac{\sin(x^{19})}{\log x}$$

(a) (7 points) Find two functions f and g such that

$$f(x) \leq \frac{\sin(x^{19})}{\log x} \leq g(x)$$

for all $x > 0$ and

$$\lim_{x\to\infty} f(x) = 0 \qquad\qquad \lim_{x\to\infty} g(x) = 0.$$

Justify your answers.

Solution. We know that $-1 \leq \sin(\theta) \leq 1$ for all θ, so

$$-\frac{1}{\log x} \leq \frac{\sin\left(x^{19}\right)}{\log x} \leq \frac{1}{\log x}$$

Let f and g be the left- and right-hand sides of this triple inequality. Then since

$$\lim_{x\to\infty} \log(x) = 0,$$

we have $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0.$ □

(b) (3 points) Finish up the proof with two magic words.

Solution. Squeeze Theorem! □

______ / 10

3. (10 Points) *Define the* signum function

$$\operatorname{sgn}(x) = \begin{cases} 0 & \text{if } x = 0; \\ \dfrac{|x|}{x} & \text{if } x \neq 0. \end{cases}$$

(i) *(2 points) What is* $\operatorname{sgn}(2)$*?* $\operatorname{sgn}(-5)$*?*

Solution. Apparently

$$\operatorname{sgn}(2) = \frac{|2|}{2} = \frac{2}{2} = 1; \operatorname{sgn}(-5) = \frac{|-5|}{-5} = \frac{5}{-5} = -1.$$

It turns out another way to write $\operatorname{sgn}(x)$ is

$$\operatorname{sgn}(x) = \begin{cases} 0 & \text{if } x = 0; \\ 1 & \text{if } x > 0; \\ -1 & \text{if } x < 0. \end{cases}$$

□

(ii) *(8 points) Where is* sgn *continuous? Use limits to justify your answer.*

Solution. For $a > 0$, we have $\operatorname{sgn}(x) = 1$ near a. Therefore

$$\lim_{x \to a} \operatorname{sgn}(x) = \lim_{x \to a} 1 = 1 = \operatorname{sgn}(a),$$

so sgn is continuous at all positive numbers. In the same manner, sgn is continuous at all negative numbers. However,

$$\lim_{x \to 0^+} \operatorname{sgn}(x) = \lim_{x \to 0^+} 1 = 1$$
$$\lim_{x \to 0^-} \operatorname{sgn}(x) = \lim_{x \to 0^-} -1 = -1,$$

so $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist, and sgn cannot be continuous at 0. □

______ / 10

4. (15 Points) *Determine the vertical and horizontal asymptotes for each of the following functions. For each vertical asymptote, also determine the left-hand and right-hand limits at that asymptote. (For example, if $x = 4$ is a vertical asymptote for f, find $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$.)*

(a) $f(x) = \dfrac{5x^2 - 45}{2x^2 - 7x + 3}$

Solution. Since f is a rational function, it may have a vertical asymptote at any point for which its denominator is 0. Since $0 = 2x^2 - 7x + 3 = (2x-1)(x-3)$, those points are $x = 1/2$ and $x = 3$.

Since

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{5x^2 - 45}{2x^2 - 7x + 3} = \lim_{x\to 3} \frac{5(x-3)(x+3)}{(2x-1)(x-3)} = \lim_{x\to 3} \frac{5(x+3)}{2x-1} = 6,$$

it turns out that $x = 3$ is a removable discontinuity and not a vertical asymptote.

On the other hand, since

$$\lim_{x\to \frac{1}{2}^-} f(x) = \lim_{x\to \frac{1}{2}^-} \frac{5(x-3)(x+3)}{(2x-1)(x-3)} = -\infty$$

and

$$\lim_{x\to \frac{1}{2}^+} f(x) = \lim_{x\to \frac{1}{2}^+} \frac{5(x-3)(x+3)}{(2x-1)(x-3)} = \infty,$$

it turns out that $x = 1/2$ is a vertical asymptote. To see that the left-hand limit holds, consider substituting a number very close to, but less than, $1/2$ into f. You'll get a very large, negative number. Similar reasoning yields the right-hand limit.

To find the function's horizontal asymptotes, we find

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \frac{5x^2 - 45}{2x^2 - 7x + 3} = \frac{5}{2}$$

and

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{5x^2 - 45}{2x^2 - 7x + 3} = \frac{5}{2}.$$

Note that the numerator and denominator of f are polynomials of the same degree, so to evaluate these limits, we merely compare the lead coefficients of the numerator and denominator. Thus f has the horizontal asymptote $y = 5/2$. □

(b) $f(x) = \dfrac{\sqrt{x^2 + 1}}{3x - 1}$

Solution. The function here is not a rational function, but the only point not in its domain is the point where its denominator equal 0, that is, the point $x = 1/3$. Since

$$\lim_{x\to\frac{1}{3}^-} f(x) = \lim_{x\to\frac{1}{3}^-} \frac{\sqrt{x^2+1}}{3x-1} = -\infty$$

and

$$\lim_{x\to\frac{1}{3}^+} f(x) = \lim_{x\to\frac{1}{3}^+} \frac{\sqrt{x^2+1}}{3x-1} = \infty,$$

it turns out that $x = 1/3$ is a vertical asymptote. Note that the numerator of f is always positive, so the sign of f depends entirely on the sign of its denominator.

To find the function's horizontal asymptotes, we find

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{3x-1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{3x-1} \cdot \frac{\sqrt{\frac{1}{x^2}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}}}{3-\frac{1}{x}} = \frac{1}{3}$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{3x-1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{3x-1} \cdot \frac{-\sqrt{\frac{1}{x^2}}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{1+\frac{1}{x^2}}}{3-\frac{1}{x}} = -\frac{1}{3}.$$

Note that $\frac{1}{x} = \sqrt{\frac{1}{x^2}}$ only when $x > 0$. When $x < 0$, $\frac{1}{x} = -\sqrt{\frac{1}{x^2}}$. Thus f has horizontal asymptotes $y = 1/3$ and $y = -1/3$.

□

(c) $f(x) = e^{2x} \cos x$

Solution. Note that $y = 2x$ is a continuous function (since it's a polynomial) and $y = e^x$ is a continuous function (since it's an exponential function), and so $y = e^{2x}$ is a continuous function (since it's the composition of two continuous functions). Also, $y = \cos x$ is a continuous function (since it's a trig function) and so $y = e^{2x} \cos x$ is a continuous function (since it's the product of two continuous functions). Thus, f is continuous everywhere and so has no vertical asymptotes.

To find the function's horizontal asymptotes, we find

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{2x} \cos x.$$

This limit does not exist since as $x \to \infty$, $\cos x$ oscillates and $e^{2x} \to \infty$. Thus f oscillates with greater and greater amplitude as $x \to \infty$.

We must also find

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^{2x}\cos x.$$

As $x \to -\infty$, $\cos x$ oscillates, but $e^{2x} \to 0$. Thus

$$\lim_{x\to-\infty} f(x) = 0.$$

Thus f has the horizontal asymptote $y = 1/3$. □

______ / 15

5. (10 Points) *Let* $f(x) = \sqrt{2x}$. *Compute* $f'(x)$ using the definition of derivative.

Solution. The slope of the line from $(x, f(x))$ to $(x+h, f(x+h)$ is

$$\begin{aligned}\frac{\sqrt{2(x+h)}-\sqrt{2x}}{h} &= \frac{\sqrt{2(x+h)}-\sqrt{2x}}{h}\cdot\frac{\sqrt{2(x+h)}+\sqrt{2x}}{\sqrt{2(x+h)}+\sqrt{2x}} \\ &= \frac{(2x+2h)-2x}{\sqrt{2(x+h)}+\sqrt{2x}} \\ &= \frac{2h}{h\left(\sqrt{2(x+h)}+\sqrt{2x}\right)} \\ &= \frac{2}{\left(\sqrt{2(x+h)}+\sqrt{2x}\right)}\end{aligned}$$

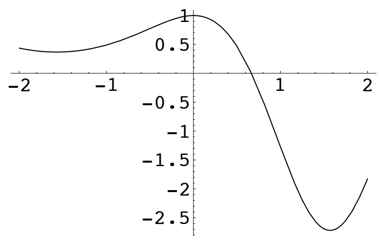
as long as $h \neq 0$. Therefore the derivative is

$$\begin{aligned}f'(x) = \lim_{h\to 0}\frac{\sqrt{2(x+h)}-\sqrt{2x}}{h} &= \lim_{h\to 0}\frac{2}{\sqrt{2(x+h)}+\sqrt{2x}} \\ &= \frac{2}{2\sqrt{2x}} = \frac{1}{\sqrt{2x}}.\end{aligned}$$

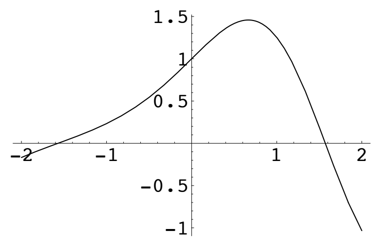
□

______ / 10

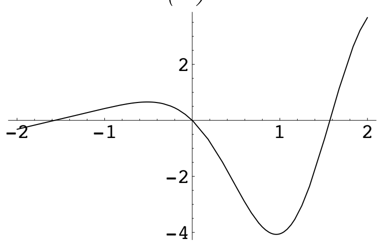
6. (8 Points) *Below are the graphs of f and its derivatives f', f'', and f'''. In the blanks below the graphs, write the letter of the graph that corresponds to each of the functions f, f', f'', and f'''.*



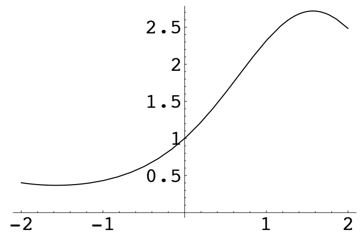
(A)



(B)



(C)



(D)

__________ f

__________ f'

__________ f''

__________ f'''

Solution. The vertical alignment helps to show C is the derivative of A, and B is the derivative of D (look at where the functions are positive/negative and increasing/decreasing). Also, A is the derivative of B. Finally, D is not the derivative of anything, because that function would always be increasing, and none of these are. So $D = f$, $B = D' = f'$, $A = B' = f''$, and $C = A' = f'''$. □

______ / 8

7. (15 Points) *Take the following derivatives.*

(i) $\frac{d}{dx}\left(x^{17}+3x^4+2x-1\right)$

Solution. We use the power rule and the sum rule:

$$\begin{aligned}\frac{d}{dx}\left(x^{17}+3x^4+2x-1\right) &= 17x^6+4\cdot 3x^3+2-0\\ &= 17x^6+12x^3+2.\end{aligned}$$

□

(ii) $\frac{d}{dx}\left(\frac{x}{1+x^2}\right)$

Solution.

$$\begin{aligned}\frac{d}{dx}\left(\frac{x}{1+x^2}\right) &= \frac{(1+x^2)(1)-x(2x)}{(1+x^2)^2}\\ &= \frac{1-x^2}{(1+x^2)^2}.\end{aligned}$$

□

(iii) $\frac{d}{dx}\left(e^x\left(7x^2+e^x\right)\right)$

Solution.

$$\begin{aligned}\frac{d}{dx}\left(e^x\left(7x^2+e^x\right)\right) &= \left(\frac{d}{dx}e^x\right)\left(7x^2+e^x\right)\\ &= e^x\frac{d}{dx}\left(7x^2+e^x\right)\\ &= e^x\left(7x^2+e^x\right)+e^x\left(14x+e^x\right))\\ &= e^x\left(e^x+14x+7x^2\right)\end{aligned}$$

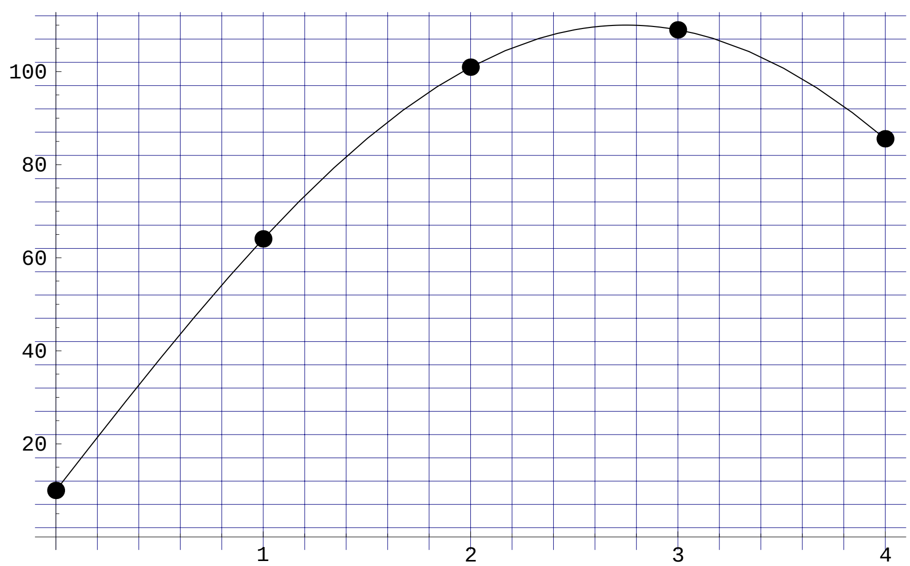
□

______ / 15

8. (10 Points) *The profits of a small company for each of the first five years of its operation are given in the following table.*

Year	*Profit in \$1000s*
2000	*10*
2001	*64*
2002	*101*
2003	*109*
2004	*86*

The company's accountant plotted points representing the profit as a function of years since 2000 and joined the points by a smooth curve. This graph is shown below.

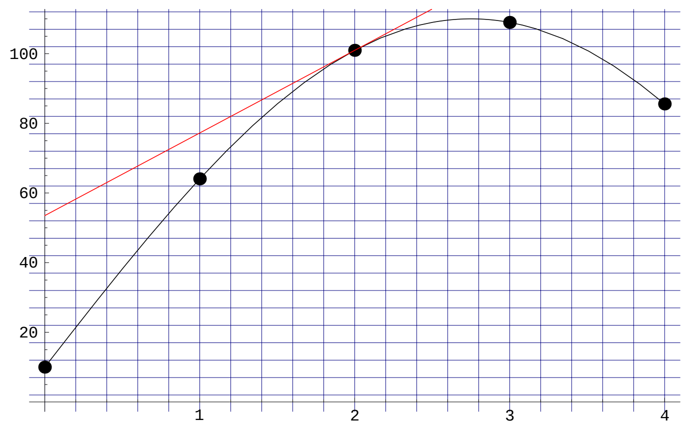


(i) *Find the slope of the secant line between the points* $(2, 101)$ *and* $(4, 86)$. *What does this slope represent in terms of profit?*

Solution. The slope of the secant line is $\dfrac{101-86}{2-4} = -\dfrac{15}{2} = -7.5$. This says that the company's profits decreased on average \$7500 per year during the period from 2002 to 2004. □

(ii) *Use the accountant's graph to estimate the rate at which profits were changing in 2002. Clearly describe the method you use to do so.*

Solution. One method to estimate the rate at which profits were changing in 2002 is to draw the tangent line to the accountant's graph at the point $(2, 101)$ and estimating the slope of this line.



The tangent line appears to have slope equal to one vertical gridline divided by one horizontal gridline. Since vertical gridlines represent a difference of \$5000 and horizontal gridlines represent a difference of .2 years, the tangent line has a slope of approximately \$25000 per year. Thus profits were increasing at a rate of \$25000 per year in 2002. □

______ / 10

9. (10 Points) *Ferdbert Freshman decides to go on a hike. On a Saturday morning he leaves at 8:00*AM *and climbs Mt. Pennypacker, arriving at 5:00*PM*. He spends the night at the top of the mountain. The next day, he climbs down the mountain, arriving back at his dorm room at 5:00*PM*.*

*Upon Ferdbert's return, his roommate Egbert (taking a break from studying his calculus) says, "Did you know that at some time today you were at the exact same elevation as you were 24 hours before?" "That's impossible," says Ferdbert. "I slept late this morning and didn't start walking until 10:00*AM*!"*

Who's right, and why?

Hint. Let f be the function which is Ferdbert's elevation at time t on Saturday, and g Ferdbert's elevation at time t on Sunday. Let $f(8{:}00\text{AM}) = 0$ and $f(5{:}00\text{PM}) = M$. What are $g(8{:}00\text{AM})$ and $g(5{:}00\text{PM})$?

Solution. We continue from the hint. $g(8{:}00\text{AM}) = M$ and $g(5{:}00\text{PM}) = 0$. Since both f and g are continuous and they switch values on the endpoints of the interval $[8{:}00\text{AM}, 5{:}00\text{PM}]$, the graphs cross by the Intermediate Value Theorem. This means there is a number t such that $f(t) = g(t)$, or a time t where Ferbert's elevation on Saturday was the same as his elevation on Sunday. □

_____ / 10

(This page intentionally left blank.)