Name: ______________________ ID#: ______________

Solutions to Midterm I

Math 1a
Introduction to Calculus

March 7, 2005

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

> *Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.*
>
> —*Handbook for Students*

1. (20 Points) *Let*

$$f(x) = \begin{cases} -\frac{1}{x} & \textit{if } x < 0; \\ x^2 - x & \textit{if } 0 \le x \le 1; \\ x - 1 & \textit{if } 1 < x < 2; \\ 0 & \textit{if } x \ge 2. \end{cases}.$$

(a) Find the left- and right-hand limits at 0, 1, *and* 2. *Justify your answers.*

Solution. All of the following limits follow from the Direct Substitution Property (page 113 of the text).

- $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 - x = 0^2 - 0 = 0$
- $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 - x = 1^2 - 1 = 0$
- $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x - 1 = 1 - 1 = 0$
- $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x - 1 = 2 - 1 = 1$
- $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} 0 = 0$

As for $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -\frac{1}{x}$, the limit laws do not apply, so we have to be a little bit more precise. I did not accept any argument using the expression "$\frac{1}{0} = \infty$"; this is ambiguous and will sometimes lead to the wrong answer. I was looking for an argument that when x is small and negative, $\frac{1}{x}$ is large and negative, while $-\frac{1}{x}$ is large and positive. Thus $\lim_{x\to 0^-} f(x) = \infty$. □

(b) At what points is f not continuous? Justify.

Solution. f is not continuous at the points where there is no limit or there is a limit which is not equal to the function value. We have no limit at 0 (because the limit from the left is infinite), nor at 2 (the right- and left-hand limits disagree). At every other point the function is continuous. □

(c) Does the graph of f have any asymptotes? Justify.

Solution. Because $\lim_{x\to 0^-} f(x) = \infty$, the vertical line $x = 0$ is an asymptote for the graph of f. Also, since both $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$, the horizontal line $y = 0$ is an asymptote for the graph of f. (Missing the horizontal asymptote was the most common mistake on this problem.) □

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2. (8 Points) *Here is a table the percentage of low-weight (< 2500 grams) births among African-American mothers in various years.*

Year	*% low birthweight*
1990	*13.25*
1991	*13.55*
1992	*13.31*
1993	*13.34*
1994	*13.24*
1995	*13.13*
1996	*13.01*
1997	*13.01*
1998	*13.05*
1999	*13.11*
2000	*12.99*
2001	*12.95*
2002	*13.29*

(i) *What is the average rate of change in the low-birthweight percentage rate over the years 1992 to 1998?*

Solution. This is a simple $\dfrac{\text{change in } y}{\text{change in } x}$ calculation.

$$\frac{\Delta \text{LBP}}{\Delta t} = \frac{13.05 - 13.31}{1998 - 1992} = -\frac{0.26}{6} \approx 0.043.$$

□

(ii) *Estimate the (instantenous) rate of change in the low-birthweight percentage rate in 1994.*

Solution. The best we can do (since we only have finitely many data points) is to take the average rate of change over [1993, 1994] and [1994, 1995] and average them. The first of these is

$$\frac{13.24 - 13.34}{1} = -0.10,$$

and the second is

$$\frac{13.13 - 13.24}{1} = -0.11.$$

The average of these is -0.105.

□

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3. (10 Points)

(i) Let f be a function. State the definition *of the derivative of f at a.*

Solution. When we ask for the definition, we are asking for something specific. As stated in Definition 2 on page 150,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

We have other notions for the derivative, such as "slope of the line tangent to the graph of f at a" or "instantaneous rate of change of f at a", but the definition is the definition. It is a limit of slopes of secant lines on the graph through $(a, f(a))$. □

(ii) Let $f(x) = \dfrac{1}{x-2}$. Use the definition of the derivative to find $f'(a)$.

Solution. We have

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h-2} - \frac{1}{a-2}}{h} \\ &= \lim_{h \to 0} \frac{(a-2) - (a+h-2)}{h(a+h-2)(a-2)} \\ &= \lim_{h \to 0} \frac{-h}{h(a+h-2)(a-2)} = \lim_{h \to 0} -\frac{1}{(a+h-2)(a-2)} = \frac{-1}{(a-2)^2}. \end{aligned}$$

□

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4. (10 Points) *Find the following derivatives. Use any method you like.*

(i) $\frac{d}{dx}4x^3 + 3x^2 + 2x + 1$

Solution. This follows from the Power Rule:

$$\begin{aligned}\frac{d}{dx}(4x^3 + 3x^2 + 2x + 1) &= 4(3x^2) + 3(2x) + 2(1)\\ &= 12x^2 + 6x + 2.\end{aligned}$$

□

(ii) $\frac{d}{dx}e^x(x-2)$.

Solution. We can use the Product Rule:

$$\begin{aligned}\frac{d}{dx}\left(e^x(x-2)\right) &= \left(\frac{d}{dx}e^x\right)(x-2) + e^x\left(\frac{d}{dx}(x-2)\right)\\ &= e^x(x-2) + e^x(1) = e^x(x-1).\end{aligned}$$

□

(iii) $f'(x)$*, where* $f(x) = \frac{2x+3}{4x-2}$. Simplify your answer.

Solution. The Quotient Rule is appropriate here:

$$\begin{aligned}f'(x) &= \frac{(4x-2)\frac{d}{dx}(2x+3) - (2x+3)\frac{d}{dx}(4x-2)}{(4x-2)^2}\\ &= \frac{(4x-2)(2) - (2x+3)(4)}{(4x-2)^2}\\ &= \frac{8x - 4 - (8x+12)}{(4x-2)^2}\\ &= \frac{-16}{2^2(2x-1)^2} = -\frac{4}{(2x-1)^2}.\end{aligned}$$

□

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5. (10 Points)

(i) State the Intermediate Value Theorem.

Solution. As in the case of the definition of the derivative, we are looking for something specific here: *Let f be continuous on the closed interval $[a, b]$, and N any number between $f(a)$ and $f(b)$. Then there exists a number c between a and b such that $f(c) = N$.*

It's really important to get the quantifiers and the order in the right order, otherwise you end up saying things that aren't necessarily true. For instance, many said something like "if c is any point between a and b, there exists N between $f(a)$ and $f(b)$ such that $f(c) = N$." But it's not true for every c that $f(c)$ lies between $f(a)$ and $f(b)$ (think about a parabola with its hump in the middle of the interval). □

(ii) Use it to prove that there exists a number c between 0 and 1 which solves the equation

$$\arctan x = 2x - \frac{\pi}{3}.$$

Solution. This was the toughest problem on the exam, and separated the A's from the A-'s. The idea is to use the IVT creatively. Notice that

$$\arctan 0 = 0 > -\frac{\pi}{3} = 2(0) - \frac{\pi}{3},$$

and

$$\arctan 1 = \frac{\pi}{4} > 2 - \frac{\pi}{3},$$

so at some point between 0 and 1 the graphs cross.

To prove this from the IVT, let

$$f(x) = \arctan x - \left(2x - \frac{\pi}{3}\right).$$

This is a continuous function since it is a difference of continuous functions. Also, $f(0) < 0$ and $f(1) > 0$. So there exists a c such that $0 < c < 1$ and $f(c) = 0$. What it means for $f(c) = 0$ is that

$$\arctan c = 2c - \frac{\pi}{3}.$$

See Figure 1. It also helps to know about the arctan function! □

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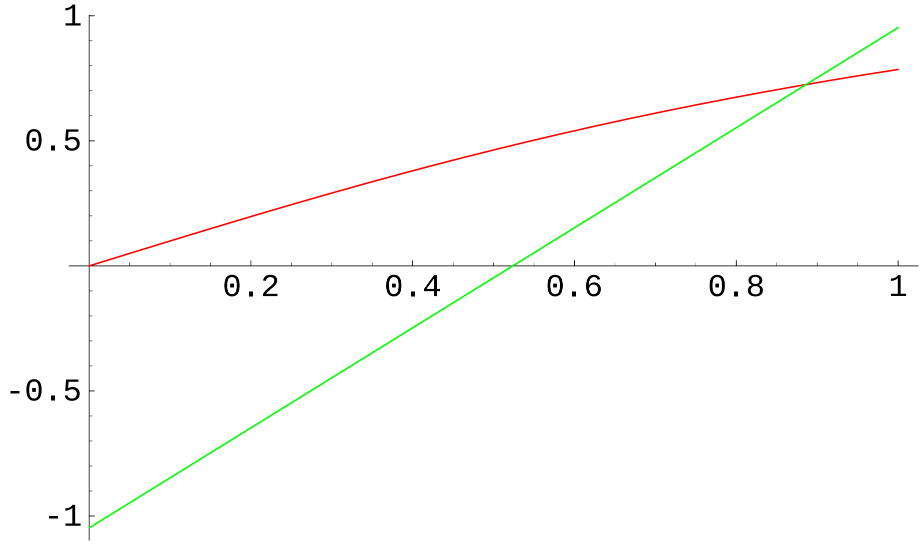

Figure 1: The graphs of arctan (red) and $2x - \frac{\pi}{3}$ (green). On one end of the interval $[0, 1]$, the red function is greater than the green one. On the other end they have the opposite order. Since both functions are continuous, they must cross at some point.