HW #21

Section 6.5 # 4, 5, 12, 13, 19, 24, 25, 27, 38

(4) $f(x) = x^3 - 3x^2 + 2x \quad [0,2]$

Verify conditions: $f(x)$ is differentiable (hence continuous) over $[0,2]$ because $f(x)$ is a polynomial.

$f(0) = 0 \quad f(2) = 8 - 12 + 4 = 0 \Rightarrow f(0) = f(2) = 0$

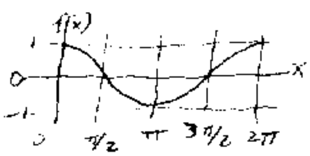
$\therefore$ The hypotheses of Rolle's Theorem are satisfied

$f'(x) = 3x^2 - 6x + 2 \qquad f'(x) = 0$ when $x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)}$

$x = \dfrac{6 \pm \sqrt{12}}{6} = \dfrac{3 \pm \sqrt{3}}{3}$ so $\underline{c = \dfrac{3+\sqrt{3}}{3} \text{ or } c = \dfrac{3-\sqrt{3}}{3}}$ (Note that both are in the interval $(0,2)$)

(5) $f(x) = \cos x \quad [\pi/2, 3\pi/2]$

Verify conditions: $f(x)$ is differentiable (hence continuous) over $[\pi/2, 3\pi/2]$ because $f(x)$ is a trig. function that's not $\tan x$.



$f(\pi/2) = 0 = f(3\pi/2)$

$f'(x) = -\sin x \qquad f'(x) = 0$ when $x = n\pi$ where n is an integer.

but $n = 1$ is the only time when $n\pi$ is in $[\pi/2, 3\pi/2]$

so $\underline{c = \pi}$

(12) $f(x) = x^3 + x - 4 \quad [-1, 2]$

Verify conditions: $f(x)$ is differentiable (hence continuous) over $[-1,2]$ because $f(x)$ is a polynomial.

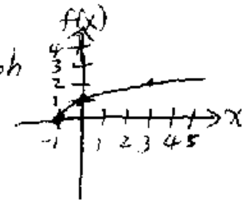
$f(-1) = -1 - 1 - 4 = -6 \quad f(2) = 8 + 2 - 4 = 6 \qquad \dfrac{f(2) - f(-1)}{2 - (-1)} = \dfrac{6 - (-6)}{3} = 4$

$f'(x) = 3x^2 + 1 = 4$ when $3x^2 = 3 \Leftrightarrow x = \pm 1$

But only $x = 1$ is in the interval $(-1, 2)$ and the MVT only covers the open interval $(-1, 2)$ for points where the average slope equals the derivative.

Therefore $\underline{c = 1}$

(13) $f(x)=\sqrt{x+1}$ $[0,3]$ Graph



It is clear from the graph that $f(x)$ is continuous and differentiable over $[0,3]$

$f(0)=1$ $f(3)=2$ $\frac{f(3)-f(0)}{3-0}=\frac{2-1}{3}=1/3$

$f'(x)=\frac{1}{2}(x+1)^{-1/2}=\frac{1}{2\sqrt{x+1}}$ $f'(x)=1/3 \Leftrightarrow \frac{1}{2\sqrt{x+1}}=1/3$

$3=2\sqrt{x+1}$

$9=4(x+1)$

$4x=5$

$x=5/4$

$5/4$ is in $(0,3)$

So $c=5/4$

(19) $f(x)=\tan x$ $(0,\pi)$

(a) $f'(x)=\sec^2 x=\frac{1}{\cos^2 x}$ $f'(x)$ is never equal to 0 because $\cos x$ cannot approach ∞ under any circumstances.

So $f'(x)\neq 0$ (and clearly $f'(c)\neq 0$) even though $f(0)=f(\pi)=0$

(b) The result in part (a) does not violate Rolle's theorem because $f(x)=\tan x$ is not continuous at $x=\pi/2$ which is within the interval $(0,\pi)$.

(24) Let 11 A.M. correspond to $x=11$ and 11 P.M. correspond to $x=23$

(a) For the purposes of this question, we can claim that the temperature fluctuated ~~continuous~~ in a continuous manner. Call the temperature-time function f.

$f(11)=76$ $f(23)=52$ So $\frac{f(23)-f(11)}{23-11}=\frac{52-76}{12}=-2$

We know that f is continuous and differentiable at all times (practical considerations) so we can apply the mean-value theorem to claim that $f'(c)=-2$ for some instant c between 11 and 23 (ie the temperature was decreasing at the rate of $2°F/h$ at c between between 11 A.M. and 11 P.M.)

(b) We know that $f(t_0)=88$ at some t_0 between 11 and 23. So $\frac{f(23)-f(t_0)}{23-t_0}=\frac{52-88}{23-t_0}=\frac{-36}{23-t_0}$

$\frac{-36}{23-t_0}$ must be less than -3 because $11<t_0<23$. Now we can apply the MVT to claim that at some instant c between t_0 and 23, $f'(c)=\frac{-36}{23-t_0}<-3$. So at some instant, the temperature was decreasing at a rate greater than $3°F/h$

(25) Let $f(t)$ be the position of runner 1
$g(t)$ ———————— runner 2

Since they both started at the same place, $f(0)=g(0)=0$. They finished in a tie so at $t=t_0$ when they finished, $f(t_0)=g(t_0)=100$. Consider the function $h(t)=f(t)-g(t)$
$h(0)=f(0)-g(0)=0-0=0$ and $h(t_0)=f(t_0)-g(t_0)=100-100=0$. Since f and g are continuous and differentiable (people run smoothly), h is continuous and differentiable. Since $h(0)=h(t_0)=0$, we can use Rolle's theorem to claim that $h'(c)=0$ at some c between 0 and t_0. So $h'(c)=f'(c)-g'(c)=0$ so $f'(c)=g'(c)$. Therefore the runners had the same velocity at least once (namely at c) during the race.

(27) (a) $f'(x)=g'(x)$ for all x in $(-\infty,\infty)$
So by the constant difference theorem, $f(x)=g(x)+K$
$f(x_0)=g(x_0)$ so $f(x_0)=g(x_0)+K$ means that $f(x_0)=f(x_0)+K$ so $K=0$
This means that $f(x)=g(x)+0$ so $f(x)=g(x)$ for all x in $(-\infty,\infty)$

(b) Let $f(x)=\sin^2x+\cos^2x$ and let $g(x)=1$

$f'(x)=2\sin x\cos x-2\cos x\sin x=0$ and $g'(x)=0$
So $f'(x)=g'(x)$ for all x. Now $f(0)=\sin^2 0+\cos^2 0=1$ and $g(0)=1$ (where 0 corresponds to x_0)
So by the result in part (a), $f(x)=g(x)$ for all x
$\therefore \sin^2x+\cos^2x=1$

(38) f and g are continuous over $[a,b]$ and differentiable over (a,b). $f(a)=g(a)$ and $f(b)=g(b)$
Consider the function $h(x)=f(x)-g(x)$. $h(x)$ is continuous over $[a,b]$ and differentiable over (a,b) because h is just the difference of f and g.
$h(a)=f(a)-g(a)=0$ and $h(b)=f(b)-g(b)=0$
So we can apply Rolle's theorem to h and claim that $h'(c)=0$ at some c in (a,b)
$h'(c)=f'(c)-g'(c)=0$ so $f'(c)=g'(c)$ at some c in (a,b)

2ND VERSION

HW # 21 / 6.5

(4) $f(x) = x^3 - 3x^2 + 2x$; $[0,2]$ $\rightarrow$ $f'(x) = 3x^2 - 6x + 2$
$f(0) = 0$, $f(2) = 0$, f is cont. and diff. on $[0,2]$

Solve $3x^2 - 6x + 2 = 0$

$$x = \frac{6 \pm \sqrt{12}}{6} = \frac{3 \pm \sqrt{3}}{3}$$

$\boxed{c = \frac{3+\sqrt{3}}{3}, \frac{3-\sqrt{3}}{3}}$ satisfy Rolle's Thm (both are on $[0,2]$)

(5) $f(x) = \cos x$; $[\pi/2; 3\pi/2]$ $\rightarrow$ $f'(x) = -\sin x$
$f(\pi/2) = 0$, $f(3\pi/2) = 0$, f is cont and diff. on $[\pi/2, 3\pi/2]$

Solve $-\sin x = 0$

$x = \pi$ is only solution on $[\pi/2, 3\pi/2]$

$\boxed{c = \pi}$

(12) $f(x) = x^3 + x - 4$; $[-1,2]$ $\rightarrow$ $f'(x) = 3x^2 + 1$
$\rightarrow$ is continuous and diff.

$$3x^2 + 1 = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{6 - -6}{3} = 4$$

$$3x^2 = 3 \longrightarrow x = \pm 1$$

$\boxed{c = 1, -1}$ satisfy Mean Value Thm

(13) $f(x) = (x+1)^{1/2}$; $[0,3] \rightarrow f'(x) = \frac{1}{2}(x+1)^{-1/2}$

continuous + diff. on $[0,3]$

$$\frac{1}{2}(x+1)^{-1/2} = \frac{f(3)-f(0)}{3} = \frac{4^{1/2}-1^{1/2}}{3} = \frac{1}{3}$$

$$(x+1)^{1/2} = 3/2$$

$$x+1 = 9/4$$

$$x = 5/4$$

$\boxed{c = 5/4}$ satisfies Mean Value Thm

(19) $f(x) = \tan x \quad (0,\pi)$

(a) $f'(x) = \sec^2 x = \frac{1}{\cos^2 x} \neq 0$ for any x

(b) $\tan x$ is not continuous on $(0, \pi)$

(24) a. avg rate of temp. change $= \frac{52°F - 76°F}{12\ h} = -2\ °F/h$

From the result of 22, there must be some instant where the temp. decreases at $2\ °F/h$

b. avg ~~te~~ rate of temp change $= \frac{52°F - 88°F}{x\ h} = \frac{-36}{x}\ °F/h$ (let x = the number of hours between when the temp is 88°F and 11 P.M.)

we know $0<x<12$, so

$$\frac{-36}{x}\ °F/h < \frac{-36}{12}\ °F/h \rightarrow \frac{-36}{x}\ °F/h < -3\ °F/h$$

From 22, we know there is some instant where temp is decreasing at $36/x\ °F/h$, which is ~~less~~ greater than $3\ °F/h$ ✓

(25) Let x_1 = position of 1 runner, x_2 = position of other runner, t = total time

Let $h = x_2 - x_1$

$$\frac{h}{t} = \frac{x_2 - x_1}{t} = 0, \text{ since both runners finished at same time}$$

From (22), ~~there~~ There is some moment where

$$\frac{d(x_2 - x_1)}{d(t)} = 0 \rightarrow \frac{dx_2}{dt} = \frac{dx_1}{dt}$$

$$v_1 = v_2$$

(27) a. $f'(x) = g'(x)$, f and g continuous on $(-\infty, +\infty)$

thus $f(x) - g(x) = k$ for all x

at x_0, $f(x_0) = g(x_0)$

So $f(x_0) - g(x_0) = 0$

so $k = 0$

$f(x) = g(x)$ ✓

b Let $f(x) = \sin^2 x$, $g(x) = 1 - \cos^2 x$

$f'(x) = 2\sin x \cos x$, $g'(x) = 2 \sin x \cos x$

$f'(x) = g'(x)$ ✓

at $x_0 = \pi/4 \rightarrow f(x_0) = \left(\frac{\sqrt{2}}{2}\right)^2 = 1/2$

$g(x_0) = 1 - \left(\frac{\sqrt{2}}{2}\right)^2 = 1/2$

$\Big\} F(x_0) = g(x_0)$

thus from a, $f(x) = g(x)$

$\sin^2 x = 1 - \cos^2 x$

$\sin^2 x + \cos^2 x = 1$ ✓

(38) f and g continuous, diff on $[a,b]$

$f(a)=g(a)$, $f(b)=g(b)$

Let $h(x)=f(x)-g(x)$, $h(x)$ is cont and diff on $[a,b]$

there is a point c, by Mean Value Thm, such that

$$h'(c)=\frac{h(b)-h(a)}{b-a}$$

$$f'(c)-g'(c)=\frac{(f(b)-g(b))-(f(a)-g(a))}{b-a}$$

$$=0$$

$$g'(c)=f'(c) \checkmark$$