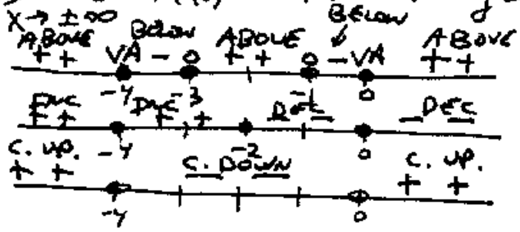
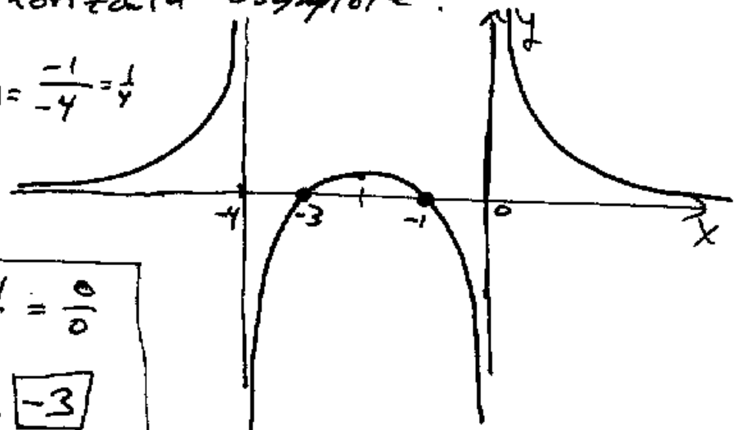
MATH 1a EXAM 2 REVIEW PROBLEM SOLUTIONS

1. $f(x)=0$ when $x^2+4x+3=(x+3)(x+1)=0 \Rightarrow x=-3,\ x=-1$. ($x$-intercepts)
$f'(x)=0$ when $-6x-12=-6(x+2)=0 \Rightarrow x=-2$ (CRIT. PT.)
$f''(x)=0$ when $3x^2+12x+16=0 \Rightarrow x=\frac{-12\pm\sqrt{144-192}}{6} \Rightarrow$ never. (NO PTS OF INFLECTION)
Vertical asymptotes at $x=0$, $x=-4$.
$\lim_{x\to\pm\infty} f(x) = +1 \Rightarrow y=1$ a horizontal asymptote.



$f(-2)=\frac{-1}{-4}=\frac{1}{4}$



2. (a) $\lim_{x\to 1}\left(\frac{x^4-5x^2+3x+1}{x^3-2x+1}\right) \to \frac{1-5+3+1}{1-2+1}=\frac{0}{0}$
$\therefore = \lim_{x\to 1}\left(\frac{4x^3-10x+3}{3x^2-2}\right)=\frac{4-10+3}{3-2}=\boxed{-3}$

(b) $\lim_{x\to 0}\frac{\sin x - x}{(\cos x-1)x} \to \frac{0}{0}$ $\quad\therefore = \lim_{x\to 0}\left(\frac{\cos x-1}{\cos x-1-x\sin x}\right)\to\frac{0}{0}$
$\therefore = \lim_{x\to 0}\left(\frac{-\sin x}{-2\sin x - x\cos x}\right)\to\frac{0}{0}$ $\quad\therefore = \lim_{x\to 0}\frac{-\cos x}{-3\cos x + x\sin x}=\frac{-1}{-3}=\boxed{\frac{1}{3}}$

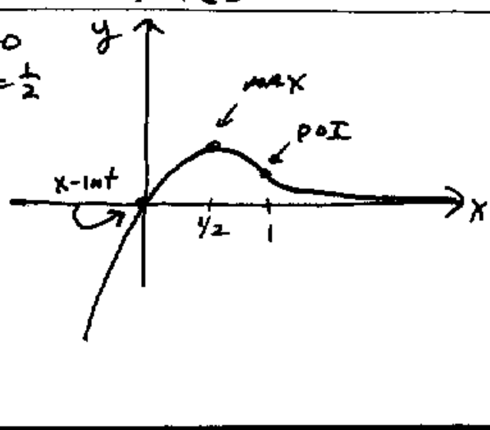
(c) LOOK AT $\lim_{x\to+\infty}\ln\left(\frac{x+4}{x-1}\right)^x = \lim_{x\to+\infty} \underbrace{x}_{+\infty}\cdot\underbrace{\ln\left(\frac{x+4}{x-1}\right)}_{0} = \lim_{x\to+\infty}\left(\frac{\ln(x+4)-\ln(x-1)}{1/x}\right)\to\frac{0}{0}$
$\therefore = \lim_{x\to\infty}\left[\frac{\frac{1}{x+4}-\frac{1}{x-1}}{(-1/x^2)}\right] = \lim_{x\to\infty} -x^2\left[\frac{(x-1)-(x+4)}{(x+4)(x-1)}\right] = \lim_{x\to+\infty}\left(\frac{5x^2}{x^2+3x-4}\right)=\boxed{5}$
$\therefore \ln[\text{LIMIT}]=5$, SO LIMIT $=\boxed{e^5}$

3. (a) $f(x)=x^4-2$ $\quad x_0=1,\ x_1=N(x_0)=x_0-\frac{f(x_0)}{f'(x_0)}=1-\frac{f(1)}{f'(1)}=1-\frac{-1}{4}=\boxed{\frac{5}{4}}$
$f'(x)=4x^3$

(b) $f(x)=x^3-7x-5$ $\quad x_0=3 \quad x_1=3-\frac{f(3)}{f'(3)}=2.95 \quad x_2=2.95-\frac{f(2.95)}{f'(2.95)}$
$f'(x)=3x^2-7$
$\cong 2.9488\ldots$

4. $f(x)=xe^{-2x}$ $\quad$ x intercept at $x=0$
$f'(x)=-2xe^{-2x}+e^{-2x}=e^{-2x}(1-2x)$ $\quad$ CRIT PT AT $x=\frac{1}{2}$
$f''(x)=-2e^{-2x}+(1-2x)e^{-2x}(-2)$ $\quad$ P.O.I at $x=1$
$= e^{-2x}(4x-4)$

x	0	$1/2$	1
$f(x)$	0	$\frac{1}{2e}$	$\frac{1}{e^2}$



(5) $g(0) = 5$, $g(2) = 11$ MVT says there's a c between 0 and 2 such that

$$g'(c) = \frac{g(2)-g(0)}{2-0} = \frac{11-5}{2} = \frac{6}{2} = 3 \text{ , i.e. slope of TL} = 3.$$

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(6) Area $A = 3\cdot y + 2\left(\tfrac{1}{2}xy\right) = 3y + xy$ and $x^2+y^2 = 9$

$$A = y(3+x) = (3+x)\sqrt{9-x^2} = A(x) \qquad 0 \le x \le 3$$

$$A'(x) = (3+x)\tfrac{1}{2}(9-x^2)^{-1/2}(-2x) + (9-x^2)^{1/2} = \frac{-x(3+x)+(9-x^2)}{\sqrt{9-x^2}} = \frac{9-3x-2x^2}{\sqrt{9-x^2}}$$

CRIT POINT when $2x^2+3x-9=0 \Rightarrow x = \frac{-3\pm\sqrt{9+72}}{4} = \frac{-3\pm\sqrt{81}}{4} = \frac{-3\pm 9}{4}$

$\Rightarrow x = \frac{3}{2}$ (in domain).

$A(0) = 9 \quad A\left(\frac{3}{2}\right) = \frac{9}{2}\sqrt{9-\frac{9}{4}} = \frac{9}{2}\sqrt{\frac{27}{4}} = \frac{27}{4}\sqrt{3} \cong 11.69$

$A(3) = 0$ So $x = 3/2$ gives MAXIMUM AREA.

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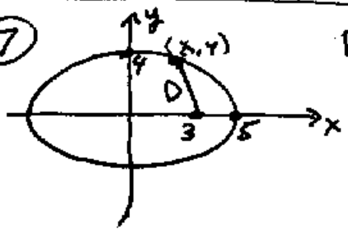
(7) 

$D^2 = (x-3)^2 + y^2$ and $\frac{x^2}{25} + \frac{y^2}{16} = 1 \Rightarrow y^2 = 16\left(1 - \frac{x^2}{25}\right)$

So $D^2 = (x-3)^2 + 16 - \frac{16}{25}x^2 = F(x) \qquad -5 \le x \le +5$

$F'(x) = 2(x-3) - \frac{32}{25}x = \frac{18}{25}x - 6 = 0 \Rightarrow x = 8\frac{1}{3}$ (OUT OF DOMAIN)

So MINIMUM DISTANCE AT ENDPOINTS $\rightarrow$ $\boxed{(5,0) \text{ clearly the closest.}}$

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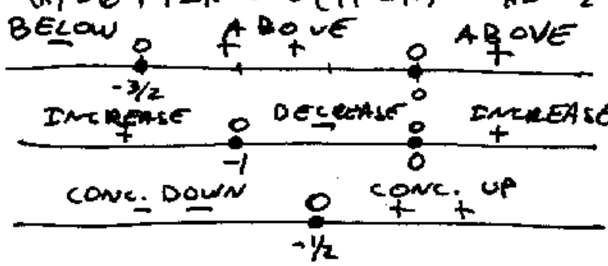
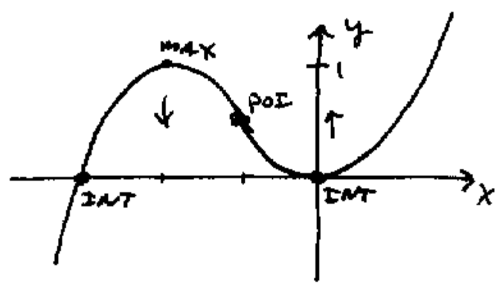
(8) $f(x) = 3x^2 + 2x^3 = x^2(3+2x)$ x-intercept at $x=0$ and $x = -3/2$ $\quad f(-1) = 1 \quad f\left(-\frac{1}{2}\right) = \frac{1}{2}$

$f'(x) = 6x + 6x^2 = 6x(1+x)$ CRIT PTS AT $x=0$, $x=-1$

$f''(x) = 6 + 12x = 6(1+2x)$ $x = -\frac{1}{2}$ possible P.O.I.





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(9) $a(t) = -t^2 \quad x(0) = 0 \quad v(0) = v_0 \quad S_{MAX} = 4 \Rightarrow v(t) = -\frac{t^3}{3} + C \Rightarrow \boxed{v(t) = v_0 - \frac{1}{3}t^3}$

$v(0) = C = v_0$

So $S(t) = v_0 t - \frac{1}{12}t^4 + C_1 \qquad \boxed{S(t) = v_0 t - \frac{1}{12}t^2}$

$S(0) = 0 - 0 + C_1 = 0 \Rightarrow C_1 = 0$

MAX DISTANCE when $v = 0 \Rightarrow v_0 - \frac{1}{3}t^3 = 0 \Rightarrow v_0 = \frac{t^3}{3}$ $\quad 3v_0 = t^3$

AT THAT TIME, $S = 4$, so $4 = v_0 t - \frac{1}{12}t^2$

Solve SIMULTANEOUSLY $\Rightarrow 4 = \frac{t^3}{3}t - \frac{1}{12}t^2 \Rightarrow 48 = 4t^4 - t^2$

$\Rightarrow 4t^4 - t^2 - 48 = 0$

$\Rightarrow t^2 = \frac{1 \pm \sqrt{1+768}}{8} \Rightarrow t^2 = \frac{1+\sqrt{769}}{8} = 3.5913\cdots$

$\therefore t \cong 1.895 \Rightarrow v_0 = 2.2686$ mi/hr.

(10) $f(x) = x^{2/3} - 2x^{1/3} = x^{1/3}(x^{1/3} - 2) = 0$ when $x=0$, $x=8$

$f'(x) = \frac{2}{3}x^{-1/3} - \frac{2}{3}x^{-2/3} = \frac{2}{3}\frac{(x^{1/3}-1)}{x^{2/3}} = 0$ when $x=1$ vert slope when $x=0$

$f''(x) = -\frac{2}{9}x^{-4/3} + \frac{4}{9}x^{-5/3} = \frac{2}{9}\frac{(-x^{1/3}+2)}{x^{5/3}} = 0$ when $x=8$, possible POI.

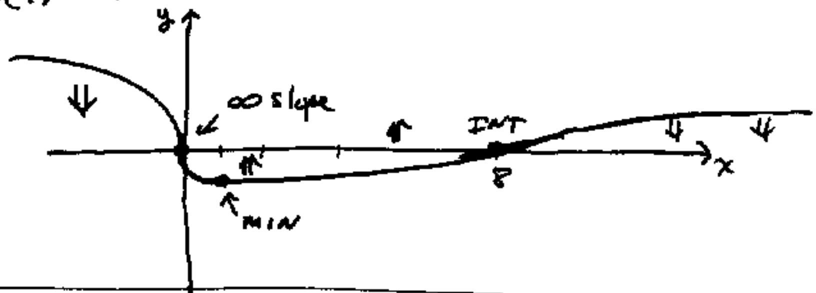
$x=0$ ALSO possible POI

ABOVE + + | 0 | BELOW − − | 8 | ABOVE + +

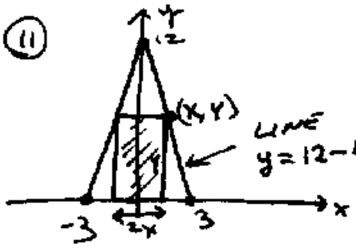
DECR | 0 | DECR ∞ − | 1 | INCR + +

CONC DOWN − − | 0 | CONC UP + + + | 8 | CONC DOWN − −

$f(1) = -1$



(11)



Area $A = (2x)y = 2x(12-4x) = 24x - 8x^2 = A(x)$

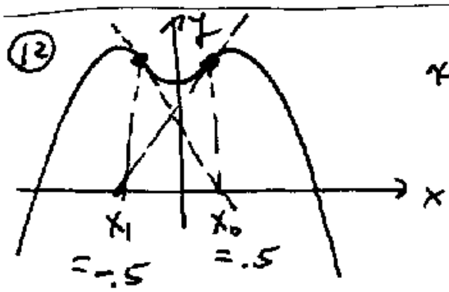
$0 \le x \le 3$ $\quad A'(x) = 24 - 16x = 0 \Rightarrow x = 3/2$

$A(0) = A(3) = 0$ $\therefore$ MAX AREA when $x = 3/2$

$A(3/2) = 36 - 8(\frac{9}{4}) = 36 - 18 = 18 = A_{MAX}$

$\boxed{A_{max} = 18 \text{ in}^2}$

(12)



$x_0 = .5 \to x_1 = -.5 \to x_2 = .5 \to x_3 = -.5$ etc

$.5 \to -.5 \to +.5 \to -.5 \to \cdots$

CYCLES BACK AND FORTH. BAD INITIAL GUESS.

Better initial guesses give roots AT $x = \pm 1.36676\ldots$

(13) Let x = # doctors hired $\quad 10000x + 4000y = 100000 \Rightarrow 5x + 2y = 50$

y = # nurses hired $\quad N = kxy = kx\left(\frac{50-5x}{2}\right) = k\left(25x - \frac{5}{2}x^2\right)$

$0 \le x \le 10$ $\quad N'(x) = k(25 - 5x) = 0 \Rightarrow x = 5$

$N(0) = 0$ $\quad N(10) = 0$ $\quad N(5) = k\left(125 - \frac{125}{2}\right) = 125k/2$

$x = \boxed{5 \text{ DOCTORS}}$ AND $y = \frac{25}{2} = \boxed{12.5 \text{ Nurses}}$.

(14)

$V = \pi r^2 h = 40$ $\quad$ Surf Area $A = 2\pi r^2 + 2\pi r h$

$h = \frac{40}{\pi r^2}$ $\quad = 2\pi r^2 + 2\pi r\left(\frac{40}{\pi r^2}\right) = 2\pi r^2 + \frac{80}{r} = A(r)$

$A'(r) = 4\pi r - \frac{80}{r^2} = 0 \Rightarrow 4\pi r = \frac{80}{r^2} \Rightarrow r^3 = \frac{20}{\pi}$

So $\boxed{r = \sqrt[3]{\frac{20}{\pi}} = 1.853 \text{ cm}}$

$\boxed{h = \frac{40}{\pi r^2} = 3.707 \text{ cm}}$

Note $\frac{h}{r} = \frac{40}{\pi r^2}\frac{1}{r} = \frac{40}{\pi r^3} = \frac{40}{\pi}\frac{\pi}{20} = 2$

$\Rightarrow$ (height = twice the radius)

15 Let x = no. of 5% reductions

price = $100000(1-.05x)$

people = $60+10x$

$\Rightarrow$ Revenue $= [100000(1-.05x)][60+10x]$

$= (100000-5000x)(60+10x)$

$\Rightarrow R(x) = 6000000 + 700000x - 50000x^2$

$R'(x) = 700000 - 100000x = 0 \Rightarrow x = 7 \Rightarrow$ <u>35% reduction.</u>

However, we must also consider the capacity of the plane!

$x=7 \Rightarrow$ people $= 60+70 = 130 \Rightarrow$ overcapacity.

$x=6 \Rightarrow$ 120 people is endpoint. $R(0) = 6$ million FF

$R(6) = 8.4$ million FF $\Leftarrow$ MAXIMUM REVENUE

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16 $v(t) = t^2 = \frac{dx}{dt}$ $\quad x(t) = \frac{1}{3}t^3 + C$

$x(2) - x(1) = (\frac{8}{3}+C) - (\frac{1}{3}+C) = \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3}}$

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17 $a(t) = -5-4t \Rightarrow v(t) = -5t - 2t^2 + C \Rightarrow x(t) = 25t - \frac{5}{2}t^2 - \frac{2}{3}t^3 + C_1$

$v(0) = 25$ ft/sec $\quad v(0) = 0 + C = 25 \quad x(0) = C_1 = 0$

$x(0) = 0$ ft. $\quad$ (a) So $\boxed{v(t) = 25 - 5t - 2t^2}$ $\quad$ So $\boxed{x(t) = 25t - \frac{5}{2}t^2 - \frac{2}{3}t^3}$

(b) BAGEL NOT MOVING when $v=0 \Rightarrow 2t^2 + 5t - 25 = 0$

$t = \frac{-5 \pm \sqrt{25+200}}{4} = \frac{-5 \pm 15}{4} \Rightarrow \boxed{t = \frac{5}{2} \text{ sec}}$

$x(0) = 0$

$x(5/2) = \frac{125}{2} - \frac{5}{2}\frac{25}{4} - \frac{2}{3}\frac{125}{8} = \frac{875}{24} = 36\ 11/24$ feet (outside $[0,2]$)

$x(2) = 50 - 10 - \frac{16}{3} = \boxed{34\ 2/3 \text{ feet} = \text{DISTANCE TRAVELLED}}$

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18 (a) $f'(x) = \frac{1}{1+\left(\frac{x-1}{x+1}\right)^2} \cdot \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2+(x-1)^2} = \frac{2}{x^2+2x+1+x^2-2x+1} = \frac{1}{x^2+1}$

<u>Note</u>: $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{x^2+1}$ also, so $\tan^{-1}\left(\frac{x-1}{x+1}\right) - \tan^{-1}(x) = $ constant

(b) $h'(x) = \frac{(x^3+2)\left[x^2\left(\frac{3}{3x+4}\right) + 2x\ln(3x+4)\right] - \left[x^2\ln(3x+4)\right](3x^2)}{(x^3+2)^2}$ $\quad$ [Note: $\ln(1) = 0$]

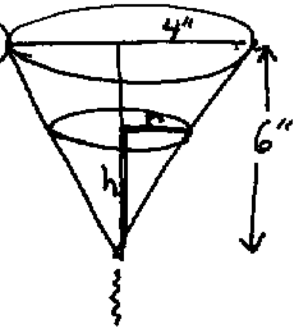
$h'(-1) = \frac{(1)[1\cdot 3 - 2\ln(1)] - [1\cdot\ln(1)](3)}{1} = \boxed{3}$

(c) $g'(t) = 3\pi t^2 + (3^{\pi t}\ln 3)\cdot\pi + \pi^3$

(d) $e^{y^2+y} = xy + 1$ $\quad$ <u>DIFF. IMPLICITLY</u>

$(e^{y^2+y})(2y+1)\frac{dy}{dx} = x\cdot\frac{dy}{dx} + y$

Plug in $x=0$, $y=-1$ $\Rightarrow -\frac{dy}{dx} = -1 \Rightarrow \boxed{\frac{dy}{dx} = 1}$

19 

SIMILAR TRIANGLES $\frac{r}{h}=\frac{4}{6} \Rightarrow 6r=4h \Rightarrow h=\frac{3}{2}r$

$V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi r^2\left(\frac{3}{2}r\right)=\frac{\pi}{2}r^3=V$

Take derivative w.r.t. time t $\Rightarrow \frac{3\pi}{2}r^2\frac{dr}{dt}=\frac{dV}{dt}$

we know that $\frac{dV}{dt}=-2\ \text{in}^3/\text{min}$. when $h=3$ in, $r=2$ in.

So $\frac{3\pi}{2}\cdot 4\frac{dr}{dt}=-2 \Rightarrow 6\pi\frac{dr}{dt}=-2 \Rightarrow \boxed{\frac{dr}{dt}=-\frac{1}{3\pi}\ \text{in/sec}}$

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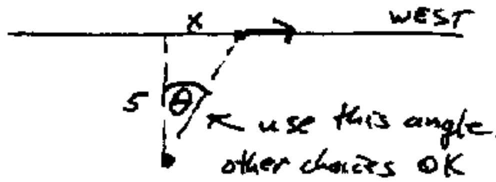
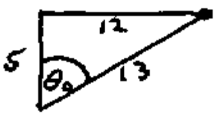
20 

use this angle. other choices OK

$\frac{dx}{dt}=507\ \text{mi/hr}$

$\frac{x}{5}=\tan\theta \Rightarrow \frac{1}{5}\frac{dx}{dt}=\sec^2\theta\frac{d\theta}{dt}$



when $x=12$ mi, $\sec\theta=\frac{13}{5}$

So $\frac{1}{5}(507)=\left(\frac{13}{5}\right)^2\frac{d\theta}{dt}=\frac{169}{25}\frac{d\theta}{dt}$

$\Rightarrow \frac{d\theta}{dt}=\frac{507}{5}\cdot\frac{25}{169}=5\cdot 3=\boxed{15\ \text{rad/hr}}$

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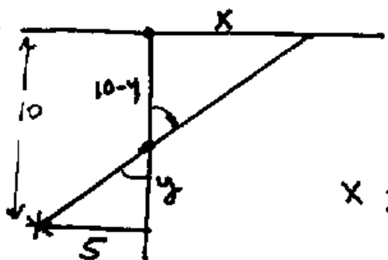
21 

Similar Triangles $\frac{x}{10-y}=\frac{5}{y}$ $\quad\frac{dy}{dt}=1\ \text{m/s}$

$\Rightarrow xy=50-5y$

$x\frac{dy}{dt}+y\frac{dx}{dt}=-5\frac{dy}{dt}$ $\quad$ PLUG IN $\frac{dy}{dt}=1\ \text{m/s}$

$y=5,\ x=5$

$\Rightarrow 5\cdot 1+5\cdot\frac{dx}{dt}=-5\cdot 1$

$\Rightarrow 5\frac{dx}{dt}=-10 \Rightarrow \frac{dx}{dt}=-2\ \text{m/s}$

(2 m/s to left)

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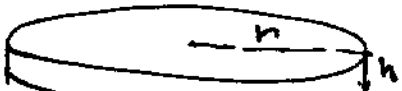
22 

$V=\pi r^2 h$ $\quad \frac{dV}{dt}=-5\ \text{ft}^3/\text{hr}$

$\frac{dV}{dt}=\pi r^2\frac{dh}{dt}+2\pi rh\frac{dr}{dt}$

$r=500$ ft, $h=.01$ ft, $\frac{dh}{dt}=-.001$ ft/hr

$\Rightarrow -5=\pi(250000)(-.001)+2\pi(500)(.01)\frac{dr}{dt} \Rightarrow \frac{dr}{dt}=\frac{250\pi-5}{10\pi}\ \text{ft/hr}$

(RADIUS INCREASING)

a

$V=A\cdot h$ $\quad \frac{dV}{dt}=A\frac{dh}{dt}+h\frac{dA}{dt} \Rightarrow -5=\pi(250000)(-.001)+(.01)\frac{dA}{dt}$

So $\frac{dA}{dt}=\frac{250\pi-5}{.01}=(25000\pi-500)\ \text{ft}^2/\text{hr}$ $\quad$ b <u>INCREASING</u>

<u>Note!</u> Could also use $A=\pi r^2$ $\quad \frac{dA}{dt}=2\pi r\frac{dr}{dt}=1000\pi\frac{dr}{dt}$ to get this.

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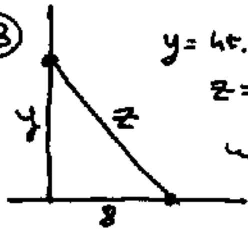

y = ht. of monkey above ground
z = length of rope
weight of monkey is irrelevant

$$64 + y^2 = z^2$$
$$2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$
$$12 \cdot 4 = 20\frac{dz}{dt}$$

$y = 6' \Rightarrow z = 10'$

$\frac{dy}{dt} = 4$ ft/sec

$$\Rightarrow \frac{dz}{dt} = \boxed{2.4 \text{ ft/sec}}$$

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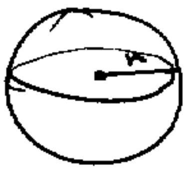
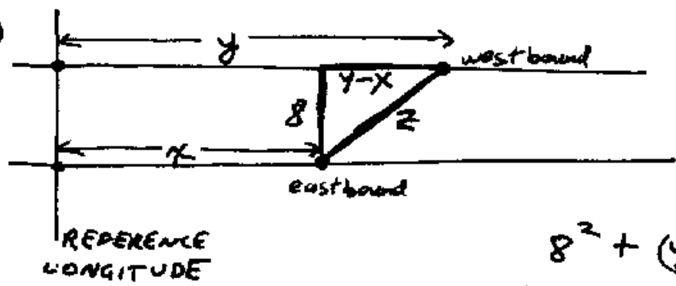


$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

If $\frac{dr}{dt} = 2$ in/min, $r = 5$ in, then $\frac{dV}{dt} = 4\pi \cdot 25 \cdot 2$

$$= \boxed{200\pi \text{ in}^3/\text{min}}$$

25



$\frac{dy}{dt} = -20$ naut mi/hr

$\frac{dx}{dt} = +20$ naut mi/hr

$$8^2 + (y-x)^2 = z^2$$
$$64 + (y-x)^2 = z^2$$

Differentiate w.r.t time $\Rightarrow$ $2(y-x)\left(\frac{dy}{dt} - \frac{dx}{dt}\right) = 2z\frac{dz}{dt}$

when ships are 10 naut mi apart, $z = 10 \Rightarrow (y-x) = 6$ mi

So $2(6)(-20-20) = 2 \cdot 10\frac{dz}{dt}$

$$\Rightarrow 12(-40) = 20\frac{dz}{dt}$$

$$\Rightarrow \frac{dz}{dt} = -\frac{480}{20} = \boxed{-24 \text{ naut mi/hr}}$$