Name of Student: ....

(Your Instructor's Name: ....)

Second Mid-Term of Math 1a

November 14, 2000 (Tuesday)
7 p.m. - 9 p.m., Science Center Hall C & E

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Question	Points	Score
1	15	
2	15	
3	12	
4	20	
5	12	
6	14	
7	12	
Total	100	

- *You have TWO hours to complete this examination.*
- *No calculators are allowed.*
- *No partial credit can be given for unsubstantiated answers.*
- *Use the back of the page if more space is needed for your answer (with an indication that your answer is continued on the back of the page).*

1. Compute

   (a) $\frac{\log_5 16 \cdot \log_2 9}{\log_5 3}$,

   Solution.

$$\frac{\log_5 16 \cdot \log_2 9}{\log_5 3} = \frac{\frac{\ln 16}{\ln 5} \cdot \frac{\ln 9}{\ln 2}}{\frac{\ln 3}{\ln 5}}$$
$$= \frac{\frac{4 \ln 2}{\ln 5} \cdot \frac{2 \ln 3}{\ln 2}}{\frac{\ln 3}{\ln 5}} = \frac{8 \ln 2}{\ln 2} = 8.$$

   (b) $\arcsin(\cos(3))$,

   Solution.

$$\arcsin(\cos 3) = \frac{\pi}{2} - \arccos(\cos 3) = \frac{\pi}{2} - 3.$$

   (c) $\cos\left(\arctan\left(\frac{5}{12}\right)\right)$.

   Solution.

   From

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1 - \cos^2 x}{\cos^2 x}$$

   it follows that

$$\cos^2 x \tan^2 x + \cos^2 x = 1$$

   and

$$\cos^2 x = \frac{1}{1 + \tan^2 x}.$$

   Thus

$$\cos x = \sqrt{\frac{1}{1 + \tan^2 x}}$$

   and

$$\cos\left(\arctan\left(\frac{5}{12}\right)\right) = \sqrt{\frac{1}{1 + \tan^2\left(\arctan\left(\frac{5}{12}\right)\right)}}$$
$$= \sqrt{\frac{1}{1 + \left(\frac{5}{12}\right)^2}} = \sqrt{\frac{1}{1 + \frac{25}{144}}}$$
$$= \sqrt{\frac{144}{169}} = \frac{12}{13}.$$

2. (a) Calculate $\frac{d}{dx}\left[\ln \arccos(x)\right]$.

Solution.

$$\frac{d}{dx}[\ln \arccos(x)] = \frac{(\arccos(x))'}{\arccos(x)} = -\frac{1}{\arccos(x)\sqrt{1-x^2}}.$$

(b) Calculate $\frac{d}{dx}\left\{[\arctan(x)]^x\right\}$.

Solution.

Since

$$\ln(\arctan(x)^x) = x\ln(\arctan(x))$$

it follows that

$$\begin{aligned}\frac{(\arctan(x)^x)'}{\arctan(x)^x} &= \ln(\arctan(x)) + x\frac{(\arctan(x))'}{\arctan(x)} \\ &= \ln(\arctan(x)) + \frac{x}{(1+x^2)\arctan(x)}\end{aligned}$$

and

$$\frac{d}{dx}[\arctan(x)^x] = \arctan(x)^x[\ln(\arctan(x)) + \frac{x}{(1+x^2)\arctan(x)}].$$

(c) Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x=0$ when x and y satisfy

$$y^3 + y\sin x - 1 = 0.$$

Solution.

We differentiate

$$y^3 + y\sin(x) - 1 = 0$$

with respect to x once and get

$$3y^2y' + y'\sin(x) + y\cos(x) = 0.$$

We differentiate again with respect to x and get

$$3y^2y'' + 6y(y')^2 + y''\sin(x) + 2y'\cos(x) - y\sin(x) = 0.$$

When $x = 0$, we have $y^3 - 1 = 0$ and $y = 1$. It follows that

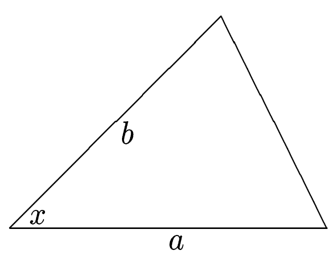
$$3 \cdot 1^2 \cdot y' + y' \cdot 0 + 1 \cdot 1 = 0.$$

and $y' = -\frac{1}{3}$ when $x = 0$. Therefore

$$3 \cdot 1^2 \cdot y'' + 6 \cdot 1 \cdot (-\frac{1}{3})^2 + y'' \cdot 0 + 2 \cdot (-\frac{1}{3}) \cdot 1 - 1 \cdot 0 = 0.$$

That is, $3y'' = 0$. Hence $y'' = 0$, when $x = 0$.

3. A triangle with sides a, b and angle x between them is evolving in time. The angle x is assumed to be acute (*i.e.*, less than a right angle). The area of the triangle remains fixed at $\frac{1}{2}$ square inches. Suppose the side a is increasing at the constant rate of 1 inch per second and the side b is increasing at the constant rate of 2 inches per second. What is the rate of change of x when $a = 1, b = \sqrt{2}$. (Hint: The area of the triangle is $\frac{1}{2}ab\sin x$.)



Solution.

The area of the triangle is $\frac{1}{2}ab\sin x$ which is kept fixed at $\frac{1}{2}$. Hence $ab\sin x = 1$. When $a = 1$ and $b = \sqrt{2}$, the equation $ab\sin x = 1$ gives $\sin x = \frac{1}{\sqrt{2}}$ and $x = \frac{\pi}{4}$.

Let t denote the time. We differentiate $ab\sin x = 1$ with respect to t and get

$$\frac{da}{dt}b\sin x + a\frac{db}{dt}\sin x + ab\cos x\frac{dx}{dt} = 0.$$

Since $\frac{da}{dt} = 1$ and $\frac{db}{dt} = 2$, it follows that at $a = 1$, $b = \sqrt{2}$, and $x = \frac{\pi}{4}$, we have

$$1 \cdot \sqrt{2} \cdot \frac{1}{\sqrt{2}} + 1 \cdot 2 \cdot \frac{1}{\sqrt{2}} + 1 \cdot \sqrt{2} \cdot \frac{1}{\sqrt{2}}\frac{dx}{dt} = 0.$$

Thus

$$\frac{dx}{dt} = -1 - \sqrt{2}$$

and the angle x is decreasing at the rate of $1 + \sqrt{2}$ radians per second.

4. (a) Compute

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}.$$

Solution.

Since $\lim_{x \to 0} \sin x - x = 0$ and $\lim_{x \to 0} x^3 = 0$, we can apply L'Hôpital's rule and get

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2}$$
$$= \lim_{x \to 0} \frac{-\sin x}{6x} = \lim_{x \to 0} \frac{-\cos x}{6} = \frac{-1}{6}.$$

(b) Compute

$$\lim_{x \to +\infty} (1 + x)^{1/x},$$

Solution.

Let

$$L = \lim_{x \to +\infty} (1 + x)^{1/x}.$$

Then

$$\ln L = \ln \left(\lim_{x \to +\infty} (1 + x)^{1/x} \right) = \lim x \to +\infty \left(\ln \left((1 + x)^{1/x} \right) \right)$$
$$= \lim x \to +\infty \frac{\ln(1 + x)}{x} = \lim x \to +\infty \frac{\frac{1}{1+x}}{1} = 0.$$

Hence $L = e^0 = 1$.

(c) Compute

$$\lim_{x \to +\infty} (1 + 1/x)^x,$$

Solution.

Note that this is our textbook's *defintion* of e. So writing down e immediately is worth full credit. However, e can be defined in several other ways, in which case we can use L'Hôpital's rule to verify that the limit is e. Let

$$L = \lim_{x\to+\infty} (1+1/x)^x .$$

Then

$$\begin{aligned}\ln L &= \ln\left(\lim_{x\to+\infty}(1+1/x)^x\right) = \lim_{x\to+\infty} \ln\left((1+1/x)^x\right) \\ &= \lim_{x\to+\infty} x\ln(1+1/x) = \lim_{x\to+\infty} \frac{\ln(1+1/x)}{\frac{1}{x}} \\ &= \lim_{x\to+\infty} \frac{\frac{1}{1/x}\frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to+\infty} \frac{1}{1+1/x} = 1.\end{aligned}$$

So $L = e^1 = e$.

(d) Compute

$$\lim_{x\to2} \frac{3^x-9}{x-2}.$$

Solution.

Since $\lim_{x\to2}(3^x-9) = 3^2-9 = 0$ and $\lim_{x\to2}(x-2) = 2-2 = 0$, we can apply L'Hôpital's rule and get

$$\lim_{x\to2} \frac{3^x-9}{x-2} = \lim_{x\to2} \frac{\frac{d}{dx}(3^x-9)}{\frac{d}{dx}(x-2)} = \lim_{x\to2} \frac{3^x \ln 3}{1} = 9\ln 3.$$

(e) Suppose $f(x)$ is a function on $(-1,1)$ and $f''(x)$ is continuous on $(-1,1)$. Suppose $f(0) = 1$, $f'(0) = 2$, and $f''(0) = 3$. Find numbers a and b such that

$$\lim_{x\to0} \frac{5f(ax)+7bf(x)-12f(0)}{x^2}$$

exists and is finite. What is the limit? (Hint: Use L'Hôpital's rule.)

Solution.

Since the limit of the denominator x^2 is 0 as $x\to0$, in order to get a finite limit, the limit of the numerator $5f(ax)+7bf(x)-12f(0)$

must be 0 as $x \to 0$. It means that $5f(0) + 7bf(0) - 12f(0)$ must be 0. Thus $5 + 7b - 12 = 0$, because $f(0) = 1 \neq 0$. It means that $b = 1$. We now set $b = 1$ and apply L'Hôpital's rule to get

$$\begin{aligned} &\lim_{x \to 0} \frac{5f(ax) + 7f(x) - 12f(0)}{x^2} \\ = &\lim_{x \to 0} \frac{\frac{d}{dx}\left(5f(ax) + 7f(x) - 12f(0)\right)}{\frac{d}{dx}\left(x^2\right)} \\ = &\lim_{x \to 0} \frac{5af'(ax) + 7f'(x)}{2x}. \end{aligned}$$

Since the limit of the denominator $2x$ is 0 as $x \to 0$, in order to get a finite limit, the limit of the numerator $5af'(ax) + 7f'(x)$ must be 0 as $x \to 0$. It means that $5af'(0) + 7f'(0)$ must be 0. Thus $5a + 7 = 0$, because $f'(0) = 2 \neq 0$. It means that $a = -\frac{7}{5}$. We now set $a = -\frac{7}{5}$ and apply L'Hôpital's rule to get

$$\begin{aligned} \lim_{x \to 0} \frac{-7f'\left(\frac{-7x}{5}\right) + 7f'(x)}{2x} &= \lim_{x \to 0} \frac{\frac{d}{dx}\left(-7f'\left(\frac{-7x}{5}\right) + 7f'(x)\right)}{\frac{d}{dx}(2x)} \\ = \lim_{x \to 0} \frac{\frac{49}{5}f''\left(\frac{-7x}{5}\right) + 7f''(x)}{2} &= \frac{\frac{49}{5}f''(0) + 7f''(0)}{2} \\ = \frac{\frac{49+35}{5}f''(0)}{2} = \frac{\frac{84}{5} \times 3}{2} &= \frac{42}{5} \times 3 = \frac{126}{5}. \end{aligned}$$

5. Find the intervals where the function

$$f(x) = 2x^3 - 9x^2 + 12x - 4$$

(a) is increasing,

(b) is decreasing,

(c) is concave up,

(d) is concave down,

and find the inflection points and relative extrema of $f(x)$. Sketch the graph of $f(x)$. How many solutions does $f(x) = 0$ have?

Solution.

From

$$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$$

it follows that

(a) $f(x)$ is increasing on the interval $(-\infty, 1]$ and also on the interval $[2, +\infty)$, and

(b) $f(x)$ is decreasing on the interval $[1, 2]$, and

$f(x)$ has a relative maximum at $x = 1$ and a relative minimum at $x = 2$ and no others.

From

$$f''(x) = 12x - 18 = 12(x - \frac{3}{2})$$

it follows that

(c) $f(x)$ is concave up on the interval $(\frac{3}{2}, +\infty)$, and

(d) $f(x)$ is concave down on the interval $(-\infty, \frac{3}{2})$, and

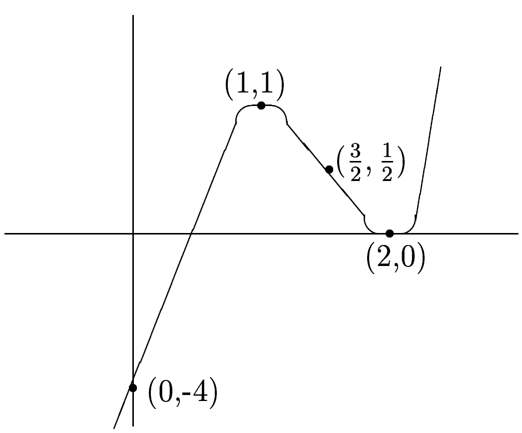
$f(x)$ has a point of inflection at $x = \frac{3}{2}$ and no others.

To plot the graph of $f(x)$ we compute its value at $x = 1, 2, \frac{3}{2}$.

$$f(1) = 2 \cdot 1^3 - 9 \cdot 1^2 + 12 \cdot 1 - 4 = 1.$$

$$f(2) = 2 \cdot 2^3 - 9 \cdot 2^2 + 12 \cdot 2 - 4 = 16 - 36 + 24 - 4 = 0.$$

$$f\left(\frac{3}{2}\right) = 2 \cdot \frac{3}{2}^3 - 9 \cdot \frac{3}{2}^2 + 12 \cdot \frac{3}{2} - 4 = \frac{27}{4} - \frac{81}{4} + 18 - 4 = \frac{-54}{4} + 14 = \frac{1}{2}.$$



There are two solutions of the equation $f(x) = 0$.

Remark. Though the problem asks only for the number of solutions of the equation $f(x) = 0$ and not the values of the solutions, yet since 2 is a double root of $f(x)$, we can actually divide $f(x)$ by $(x-2)^2$ and get $f(x) = (2x-1)(x-2)^2$ to find out that the two zeroes are $x = \frac{1}{2}$ and $x = 2$.

6. (a) Find the smallest and largest values that the function

$$f(x) = x^3 - 3x + 1$$

takes on the interval $[0, 2]$.

Solution.

Since $f(x) = x^3 - 3x + 1$, to find the maximum and minimum of that function we evaluate $f(x)$ at all its critical points in the interval $[0, 2]$, as well as at the endpoints of that interval.

Observe that $f(x) = x^3 - 3x + 1$ is differentiable, so that all critical points of $f(x)$ are points where $f'(x) = 0$. Since $f'(x) = 3x^2 - 3$, the solutions of $f'(x) = 0$ have $x^2 = 1$, and so $x = \pm 1$. However, $x = -1$ is *not* in the interval $[0, 2]$, and so is not a relevant critical point.

Therefore, we test $f(1) = 1-3+1 = -1$, $f(0) = 0-0+1 = 1$, and $f(2) = 8-6+1 = 3$. Hence $f(1) = -1$ is the absolute minimum of $f(x)$ on the given interval, and $f(2) = 3$ is the absolute maximum.

(b) Consider the function

$$f(x) = \arctan(x) + \frac{4}{x+2}$$

defined for $-2 < x < +\infty$.

(i) Find all the points $-2 < x < +\infty$ where the relative extrema of f occur.

Solution.

$$f'(x) = \frac{1}{1+x^2} - \frac{4}{(x+2)^2} = \frac{(x+2)^2 - 4(1+x^2)}{(1+x^2)(x+2)^2}$$
$$= \frac{-3x^2+4x}{(1+x^2)(x+2)^2} = \frac{-3x\left(x-\frac{4}{3}\right)}{(1+x^2)(x+2)^2}.$$

Thus the point $x = 0$ is a relative minimum and the point $x = \frac{4}{3}$ is a relative maximum for $f(x)$.

(ii) Find $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-2^+} f(x)$.

Solution.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(\arctan(x) + \frac{4}{x+2}\right) = \frac{\pi}{2}.$$
$$\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+}\left(\arctan(x) + \frac{4}{x+2}\right) = +\infty.$$

(iii) Use (i) and (ii) to find all the points $-2 < x < +\infty$ where the absolute extrema of f occur.

Solution.

The values of $f(x)$ at $x = 0$ and at $x = \frac{4}{3}$ are given by

$$f(0) = 2.$$
$$f\left(\frac{4}{3}\right) = \arctan\left(\frac{4}{3}\right) + \frac{4}{\frac{4}{3}+2} = \arctan\left(\frac{4}{3}\right) + \frac{6}{5}.$$

We compare the values of

$$\lim_{x\to+\infty} f(x) = \frac{\pi}{2},$$
$$\lim_{x\to-2^+} f(x) = +\infty,$$
$$f(0) = 2 > \frac{\pi}{2},$$
$$f\left(\frac{4}{3}\right) = \arctan\left(\frac{4}{3}\right) + \frac{6}{5} > \frac{\pi}{4} + \frac{6}{5} > \frac{\pi}{2}.$$

The maximum is $+\infty$ and the minimum is $\frac{\pi}{2}$. However, both values are the limits of the functions at the end-points of the interval $(-2, +\infty)$. The end-points $x = 2$ and $x = +\infty$ are not themselves in the interval $(-2, +\infty)$. Hence no absolute maximum and no absolute minimum are achieved by the function $f(x)$ on the interval $(2, +\infty)$.

7. Lara is driving due east at 15 meters per second and Lauren is driving due north at 20 meters per second. They are approaching the same intersection: in fact, when Lara is 40 meters away from the intersection, Lauren is 45 meters away from it.

   (a) If they continue at the same velocities, what will be the minimum distance between the two cars as they pass through the intersection?

   (b) How fast are they moving away from each other at that time? (*i.e.,* What is the rate of change of their distance when their distance assumes its minimum value?)

Solution.

(a) We assume the roads are straight and intersect at a right angle. Introduce a coordinate system where the origin is at the intersection, Lara is driving along the x-axis in the positive direction, and Lauren is driving along the y-axis in the positive direction. Let x and y be the coordinates of Lara and Lauren, respectively, and D the distance between them. Let $t = 0$ be the time when $x = -40$ and $y = -45$. Then x, y, D satisfy

$$x = 15t - 40, \qquad y = 20t - 45,$$
$$D^2 = x^2 + y^2 = (15t - 40)^2 + (20t - 45)^2.$$

When D achieves its smallest value, $\frac{1}{2}D^2$ also achieves its smallest value, hence

$$0 = \frac{d}{dt}(\frac{1}{2}D^2) = 15(15t - 40) + 20(20t - 45),$$

which gives $t = 12/5$. At this time,

$$x = 15 \cdot (\frac{12}{5}) - 40 = -4, \qquad y = 20 \cdot (\frac{12}{5}) - 45 = 3,$$

so the minimum distance between Lara and Lauren is

$$D_{\text{min}} = \sqrt{(-4)^2 + 3^2} = 5.$$

(b) At the time when D is smallest, $\frac{dD}{dt} = 0$. There is no need to do any computations here, since this is a general fact about derivatives at extreme points.