Name: ______________________________ ID#: ____________________

Solutions to
Midterm Examination I

Math 1a
Introduction to Calculus

12 March 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

> *Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.*
>
> —*Handbook for Students*

Problem Number	Possible Points	Points Earned
1	12	
2	12	
3	20	
4	21	
5	12	
6	9	
7	14	
Total	100	

1. (12 Points) *Find the following:*

(i) $\log_3 243$

Solution. Since $3^5 = 243$, the answer is 5. □

(ii) $\log_2 \frac{1}{64}$

Solution. Since $2^6 = 64$, the answer is -6. □

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2. (12 Points) *Let* $f(x) = \sqrt{4 - 3x^2}$. *Find the domain and range of* f. *Explain your answers!*

Solution. We must eliminate the numbers which make the expression under the square root sign negative. That is, the domain is restricted to those numbers such that

$$\begin{aligned} &4 - 3x^2 \geq 0 \\ \implies\ &x^2 \leq \frac{4}{3} \\ \implies\ &|x| \leq \frac{2}{\sqrt{3}}. \end{aligned}$$

So the domain is $\left[-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right]$. A very common mistake was to forget the absolute value and result in a domain of $\left(-\infty, \frac{2}{\sqrt{3}}\right]$.

In computing the range, we know that square roots give only nonnegative numbers, and the largest the expression under the radical (this is called the *radicand* if you care) can be is if $x = 0$, i.e., 4. Thus the range is $[0, 2]$. The most common incorrect answer was $[0, \infty)$. This is the range of the square root function, but we only only plugging in numbers between 0 and 4! □

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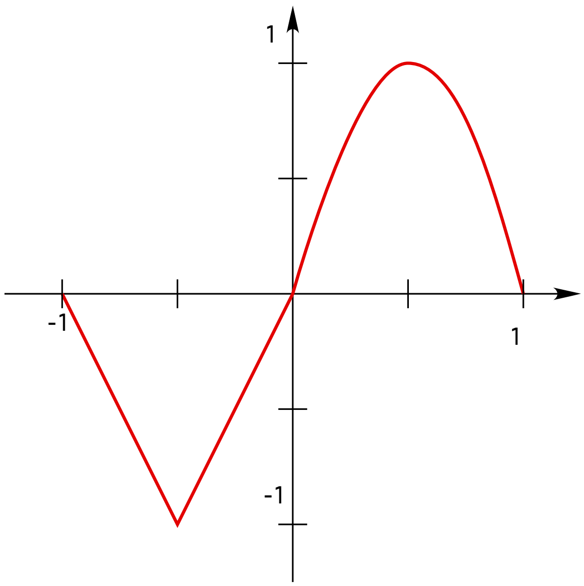
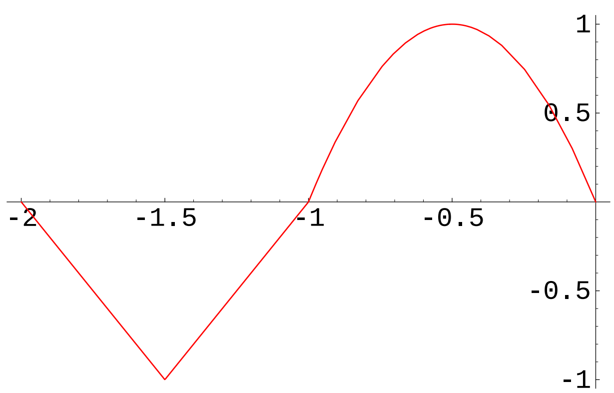
3. (20 Points) *Depicted is the graph of a function f.*



Draw the graphs of the following functions, labeling the endpoints of the domain as well as the maximum and minimum points.

(i) $g(x) = f(x+1)$

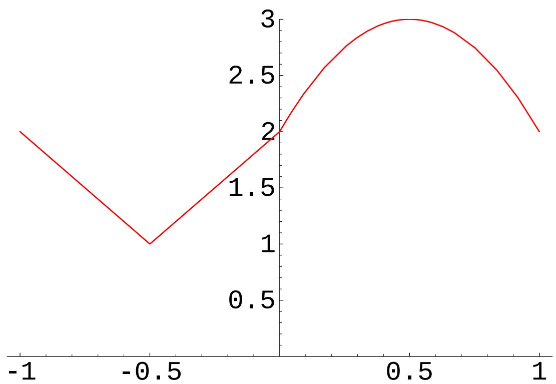
Solution. The graph of g is that of f translated to the *left* by 1 unit.



□

(ii) $h(x) = f(x) + 2$

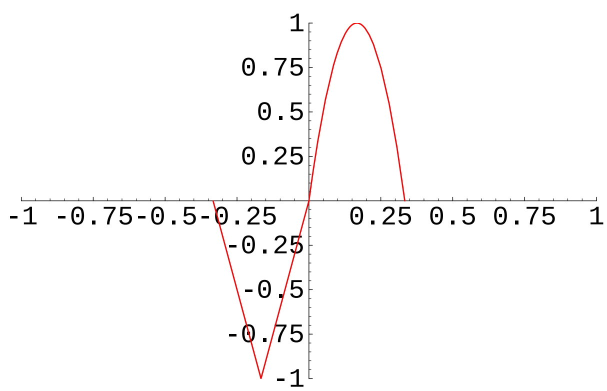
Solution. The graph of h is that of f translated *up* by 2.



□

(iii) $k(x) = f(3x)$

Solution. This turned out to be one of the trickier ones. The graph of k is that of f *compressed* by a factor of 3 in the horizontal direction. To see why this is, consider $k(1) = f(3)$? We don't know what $f(3)$. But $k(1/6) = f(3 \cdot 1/6) = f(1/2) = 1$.

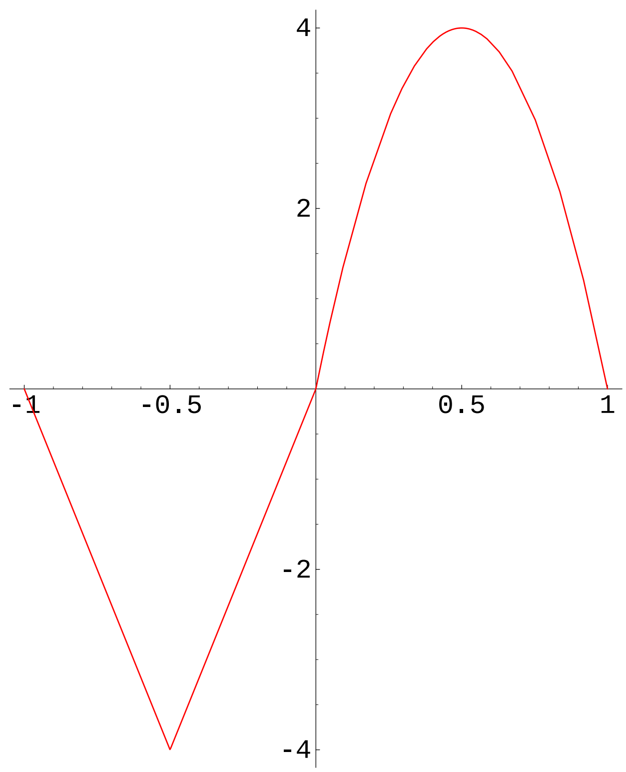


□

(iv) $\ell(x) = 4f(x)$.

Solution. The graph of k is that of f dilated (stretched) by a factor of 4 in the vertical direction.

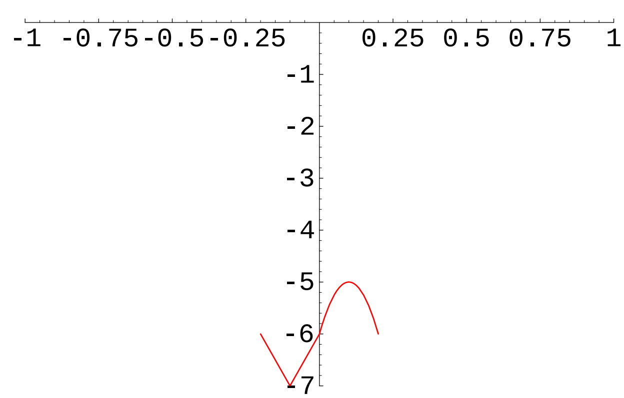
3 3



□

(v) $m(x) = f(5x) - 6.$

Solution. The graph of k is that of f compressed by a factor of 5 in the horizontal direction, then translated 6 downward.



□

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4. (21 Points) *Define*

$$f(x) = \begin{cases} x+4 & x \le 0; \\ (x-2)^2 & 0 < x \le 2; \\ 4 & x > 2. \end{cases}$$

(a) Find the following:

(i) $\lim_{x\to 0^+} f(x)$

Solution. Remember that polynomials are continuous, so

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x-2)^2 = (0-2)^2 = 4.$$

□

(ii) $\lim_{x\to 0^-} f(x)$

Solution.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x+4 = 0+4 = 4.$$

□

(iii) $\lim_{x\to 0} f(x)$

Solution. Since the limit from both sides is 4, we have $\lim_{x\to 0} f(x) = 4$. □

(iv) $\lim_{x\to 2^-} f(x)$

Solution.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x-2)^2 = (2-2)^2 = 0.$$

□

(v) $\lim_{x\to 2^+} f(x)$

Solution.

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 4 = 4.$$

□

(vi) $\lim_{x\to 2} f(x)$

Solution. Since the limits from either side do not agree, the limit does not exist. □

(b) At which points is f discontinuous? Why?

4

4

Solution. The function is discontinuous at 2 because although the function is defined there, the limit $\lim_{x\to 2} f(x)$ does not exist. This is the real, mathematical answer to "Why?" We speak colloquially about being able to draw the graph without picking up the pencil; that is good intuition but is imprecise. □

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5. (12 Points) *Assume the limit* $\lim_{h\to 0} \frac{2^h-1}{h}$ *exists. Call this number* r.

Let $f(x) = 2^x$. *Use the definition of the derivative to determine* $f'(a)$ *(your answer will involve* r*).*

Solution. We have

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{2^{a+h} - 2^a}{h} \\ &= \lim_{h\to 0} \frac{2^a 2^h - 2^a}{h} \\ &= \lim_{h\to 0} 2^a \frac{2^h - 1}{h} \\ &= 2^a \lim_{h\to 0} \frac{2^h - 1}{h} = 2^a \cdot r. \end{aligned}$$

□

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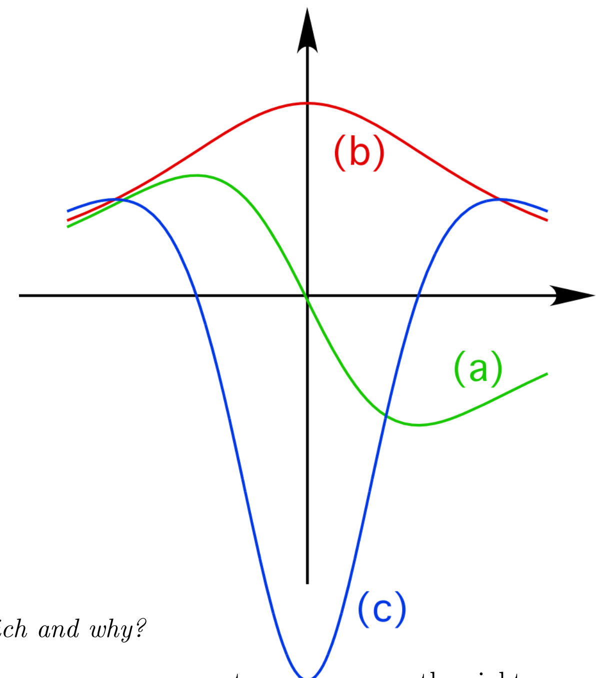
6. (9 Points) *Below are graphed three functions. One of them is a function f, another is f', and the third is f''.*



Which is which and why?

Solution. There are many ways to come across the right answer, lucky guess being one of them. Notice first that (b) is not the derivative of anything shown, since it's always positive and the antiderivative would therefore always be increasing. None of the functions pictured have that property, so (b) must be f.

To find f', we need a function which has a zero at 0, is positive for $x < 0$, and negative for $x > 0$. (a) fits this description. This leaves (c) for f''. $\square$

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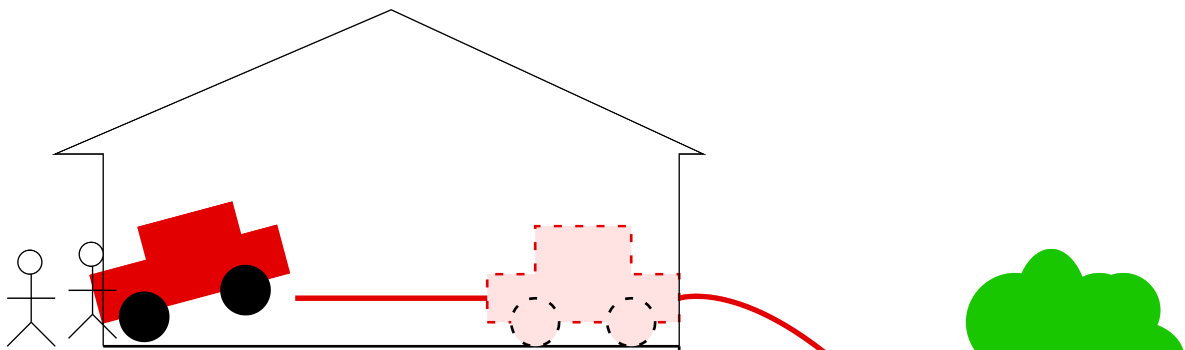
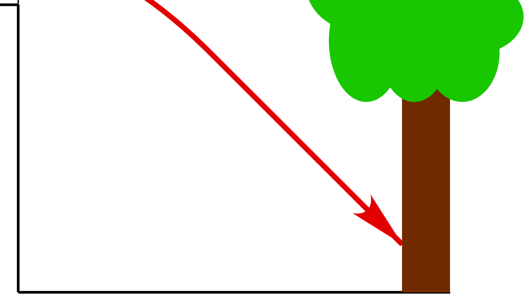
7. (14 Points)

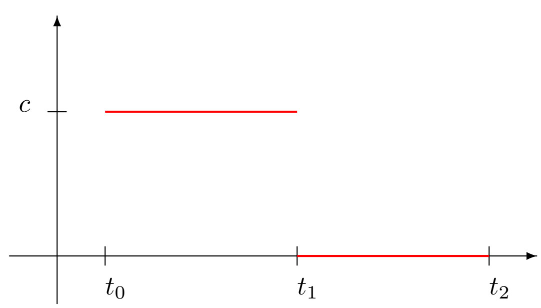




Ferris and Cameron have Cameron's father's car up on blocks. They are trying to roll back its odometer by running the car in reverse while the wheels spin. At time t_0, Cameron kicks the car off the blocks in a fit of pent-up rage against his father. It hits the ground, whereupon it speeds through the garage, accelerating at a constant rate. At time t_1, the car breaks through the back wall of the garage, then falls (accelerated by gravity) to the ravine below. At time t_2, the car crashes into the ground.

(a) Graph the acceleration of the car in the horizontal *direction over the time interval* $[t_0, t_2]$.

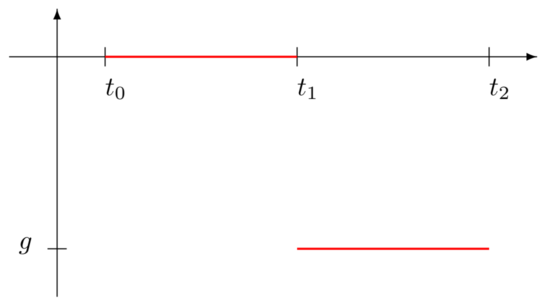
Solution. Notice that the car accelerates at a constant rate (call it c; we will orient things so that right is positive even though the car is going backwards!) until crashing through the wall. After that point, the wheels are no longer touching the ground, so there is no more horizontal acceleration. Thus the graph of acceleration versus time looks like



□

(b) Graph the acceleration of the car in the vertical *direction over the same time interval.*

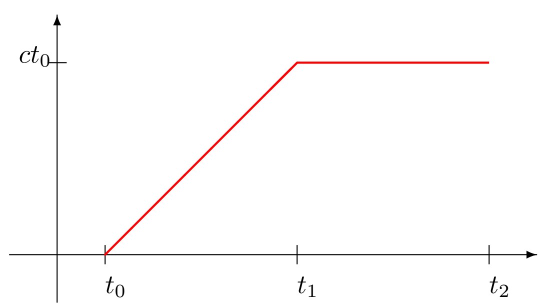
Solution. I should have written that Cameron kicks the car and it hits the ground at time t_0. I wasn't interested in the acceleration of the car as it fell off the blocks. On the interval $[t_0, t_1]$, the car is on the ground and thus has no vertical acceleration. Once the car crashes through the wall at time t_1, the car is accelerated downward by gravity at a constant rate g. Thus the graph looks like:



□

(c) *Graph the velocity of the car in the* horizontal *direction.*

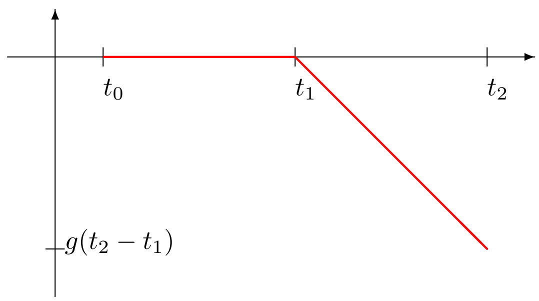
Solution. The important thing is that acceleration is the rate of change of velocity. So this graph should be the anitderivative of the graph in (a). I tried to grade this problem assuming your answer in (a) is right. In our case, we see that the velocity increases at a constant rate and then ceases to increase at time t_1. Thus the horizontal velocity looks like



□

(d) *Graph the velocity of the car in the* vertical *direction.*

Solution. In this case, there is no vertical velocity until t_1, after which time the velocity decreases at a constant rate. Thus:



□

□ *Check the box if you know what movie this problem comes from.*

71% of the class has seen the great American movie *Ferris Bueller's Day Off*.

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