

4.5

Note: The use of l'Hospital's Rule is indicated by an H above the equal sign: $\overset{H}{=}$

1. (a) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$.

(b) $\lim_{x \rightarrow a} \frac{f(x)}{p(x)} = 0$ because the numerator approaches 0 while the denominator becomes large.

(c) $\lim_{x \rightarrow a} \frac{h(x)}{p(x)} = 0$ because the numerator approaches a finite number while the denominator becomes large.

(d) If $\lim_{x \rightarrow a} p(x) = \infty$ and $f(x) \rightarrow 0$ through positive values, then $\lim_{x \rightarrow a} \frac{p(x)}{f(x)} = \infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x^2$.] If $f(x) \rightarrow 0$ through negative values, then $\lim_{x \rightarrow a} \frac{p(x)}{f(x)} = -\infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = -x^2$.] If $f(x) \rightarrow 0$ through both positive and negative values, then the limit might not exist. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x$.] It is not possible to evaluate this limit.

(e) $\lim_{x \rightarrow a} \frac{p(x)}{q(x)}$ is an indeterminate form of type $\frac{\infty}{\infty}$.

4. (a) $\lim_{x \rightarrow a} [f(x)]^{p(x)}$ is an indeterminate form of type 0^0 .

(b) If $y = [f(x)]^{p(x)}$, then $\ln y = p(x) \ln f(x)$. When x is near a , $p(x) \rightarrow \infty$ and $\ln f(x) \rightarrow -\infty$, so $\ln y \rightarrow -\infty$. Therefore, $\lim_{x \rightarrow a} [f(x)]^{p(x)} = \lim_{x \rightarrow a} y = \lim_{x \rightarrow a} e^{\ln y} = 0$, provided f^p is defined.

(c) $\lim_{x \rightarrow a} [h(x)]^{p(x)}$ is an indeterminate form of type 1^∞ .

(d) $\lim_{x \rightarrow a} [p(x)]^{f(x)}$ is an indeterminate form of type ∞^0 .

(e) If $y = [p(x)]^{q(x)}$, then $\ln y = q(x) \ln p(x)$. When x is near a , $q(x) \rightarrow \infty$ and $\ln p(x) \rightarrow \infty$, so $\ln y \rightarrow \infty$. Therefore, $\lim_{x \rightarrow a} [p(x)]^{q(x)} = \lim_{x \rightarrow a} y = \lim_{x \rightarrow a} e^{\ln y} = \infty$.

(f) $\lim_{x \rightarrow a} \sqrt[q(x)]{p(x)} = \lim_{x \rightarrow a} [p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0 .

9. This limit has the form $\frac{0}{0}$. $\lim_{x \rightarrow 0} \frac{\tan px}{\tan qx} \overset{H}{=} \lim_{x \rightarrow 0} \frac{p \sec^2 px}{q \sec^2 qx} = \frac{p(1)^2}{q(1)^2} = \frac{p}{q}$

12. $\lim_{x \rightarrow \infty} \frac{\ln \ln x}{x} \overset{H}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x \ln x} = 0$

27. $\lim_{x \rightarrow 0} \left(\frac{1}{x} - \csc x \right) = \lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right) = \lim_{x \rightarrow 0} \frac{\sin x - x}{x \sin x}$
 $\overset{H}{=} \lim_{x \rightarrow 0} \frac{\cos x - 1}{x \cos x + \sin x} \overset{H}{=} \lim_{x \rightarrow 0} \frac{-\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0$

33. $y = (1 - 2x)^{1/x} \Rightarrow \ln y = \frac{1}{x} \ln(1 - 2x)$, so $\lim_{x \rightarrow 0} \ln y = \lim_{x \rightarrow 0} \frac{\ln(1 - 2x)}{x} \overset{H}{=} \lim_{x \rightarrow 0} \frac{-2/(1 - 2x)}{1} = -2 \Rightarrow$
 $\lim_{x \rightarrow 0} (1 - 2x)^{1/x} = \lim_{x \rightarrow 0} e^{\ln y} = e^{-2}$.

54. (a) $\lim_{t \rightarrow \infty} v = \lim_{t \rightarrow \infty} \frac{mg}{c} (1 - e^{-ct/m}) = \frac{mg}{c} \lim_{t \rightarrow \infty} (1 - e^{-ct/m})$
 $= \frac{mg}{c} (1 - 0)$ [because $-ct/m \rightarrow -\infty$ as $t \rightarrow \infty$] $= \frac{mg}{c}$,

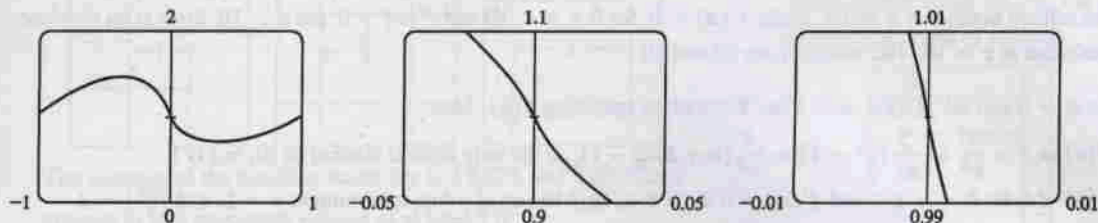
which is the speed the object approaches as time goes on, the so-called limiting velocity.

58. (a) For f to be continuous, we need $\lim_{x \rightarrow 0} f(x) = f(0) = 1$. We note that for $x \neq 0$, $\ln f(x) = \ln |x|^x = x \ln |x|$.

So $\lim_{x \rightarrow 0} \ln f(x) = \lim_{x \rightarrow 0} x \ln |x| = \lim_{x \rightarrow 0} \frac{\ln |x|}{1/x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{1/x}{-1/x^2} = 0$. Therefore,

$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} e^{\ln f(x)} = e^0 = 1$. So f is continuous at 0.

(b) From the graphs, it appears that f is differentiable at 0.



(c) To find f' , we use logarithmic differentiation: $\ln f(x) = x \ln |x| \Rightarrow \frac{f'(x)}{f(x)} = x \left(\frac{1}{x} \right) + \ln |x| \Rightarrow$

$f'(x) = f(x)(1 + \ln |x|) = |x|^x(1 + \ln |x|)$, $x \neq 0$. Now $f'(x) \rightarrow -\infty$ as $x \rightarrow 0$ [since $|x|^x \rightarrow 1$ and $(1 + \ln |x|) \rightarrow -\infty$], so the curve has a vertical tangent at $(0, 1)$ and is therefore not differentiable there. The fact cannot be seen in the graphs in part (b) because $\ln |x| \rightarrow -\infty$ very slowly as $x \rightarrow 0$.