Name: Karen J. Ng

Mathematics 1a
Final Examination
May 24, 1995

Please circle the name of your section leader:

Srdjan Divac   Robert Kaplan (10:00)   Robert Kaplan (11:00)   Esther Silberstein

Show all your work.

Question	Points	Score
1	6	6
2	6	5
3	8	6
4	6	6
5	7	7
6	8	8
7	9	7
8	9	9
9	10	8
10	10	10
11	8	6
12	13	13
Total	100	91

1. (6 points) Use the definition of derivative to find $f'(x)$ if $f(x) = \frac{1}{1-x}$.

$$\frac{0 \cdot f - 1(-1)}{(1-x)^2}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{\cancel{x} + h - \cancel{x}}$$

$$= \frac{\frac{1}{1-(x+h)} - \frac{1}{1-x}}{h}$$

$$= \frac{\frac{1}{1-x-h}^{(1-x)} - \frac{1}{1-x}}{h}$$

$$= \frac{\frac{1-x}{(1-x-h)(1-x)} - \frac{1-x-h}{(1-x)(1-x-h)}}{h}$$

$$= \frac{\cancel{1} - \cancel{x} - \cancel{1} + \cancel{x} + \cancel{h}}{\cancel{h}(1-x)(1-x-h)}$$

$$f'(x) = \lim_{h \to 0} \frac{1}{(1-x)(1-x-\underset{\downarrow \, 0}{h})}$$

$$f'(x) = \frac{1}{(1-x)^2}$$

2. (6 points) Find the derivatives of the following functions:

(a) $g(x) = x^3 \cdot e^x$

$g'(x) = 3x^2 \cdot e^x + x^3 \cdot e^x$

$g'(x) = e^x (3x^2 + x^3)$

✓

(b) $k(x) = \ln(\sin x + \cos x)$

$K'(x) = \frac{1}{(\sin x + \cos x)} \cdot (\cos x - \sin x)$

$K'(x) = \frac{\cos x - \sin x}{\sin x + \cos x}$

✓

(c) $f(x) = \frac{\arcsin(\sqrt{x})}{1+x}$ $\frac{f(x)}{g(x)}$ $\frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$ $\sin^{-1} \xrightarrow{\text{derivative}} \frac{1}{\sqrt{1-x^2}}$

$f'(x) = \frac{\frac{1}{\sqrt{1-(\sqrt{x})^2}}(1+x) - (1)(\sin^{-1}\sqrt{x})}{(1+x)^2}$

~1

$f'(x) = \frac{\frac{(1+x)}{\sqrt{1-x}} - \sin^{-1}\sqrt{x}}{(1+x)^2}$

3

3. **(8 points) Find antiderivatives of the following functions:**

(a) $f(t) = 2t^7 - 13t + \dfrac{3}{\cos^2 t}$

$F(t) = \frac{1}{4}t^8 - 6.5t^2 + -3\cdot(\cos t)^{-1} = \boxed{\frac{1}{4}t^8 - 6.5t^2 - 3(\cos t)^{-}}$

-1

$2\frac{8}{7}t^7 - 13t + 3(\cos t)^{-2}$

(b) $h(t) = \dfrac{(\ln 3t)^3}{t}$

-1 $\quad H(t) = \frac{1}{4}(\ln 3t)^4 \cdot 3 = \boxed{\frac{3}{4}(\ln 3t)^4}$

$3/(\ln 3t)^3 \quad \frac{1}{3t}$

(c) $g(t) = \dfrac{1-3t+t^2}{t^2} = \frac{1}{t^2} - \frac{3\cancel{t}}{\cancel{t^2}} + \frac{t^2}{t^2} = t^{-2} - 3t^{-1} + 1$

$\checkmark$ $\boxed{G(t) = -t^{-1} - 3\ln t + t}$ $\quad -\frac{3}{t}$

$+1t^{-2} - 3\frac{1}{t} + 1 \qquad \frac{1}{t^2} - \frac{3}{t} + \frac{t^2}{t^2} \qquad \frac{1-3t+t^2}{t^2}$

(d) $m(t) = \dfrac{e^t}{\sqrt{1-e^{2t}}} = e^t \cdot \frac{1}{\sqrt{1-(e^t)^2}}$

$\checkmark$ $\boxed{M(t) = \sin^{-1} e^t}$

$= \frac{1}{\sqrt{1-(e^t)^2}} \cdot e^t$

$\tan x = \frac{\cos y}{y}$ $\quad \frac{-\sin y \frac{dy}{dx}(y) - \cos}{y^2}$

$\sec^2 x =$

$y^2 \sec^2 x \mp \cos y = d$

$-\sin y \ (y) \quad d$

4. **(6 points) Given the curve $y \tan x = \cos y$,**

(a) find $\frac{dy}{dx}$

$$\frac{dy}{dx} \cdot \tan x + y \cdot \sec^2 x = -\sin y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} \cdot \tan x + \sin y \cdot \frac{dy}{dx} = -y \cdot \sec^2 x$$

$$\frac{dy}{dx}(\tan x + \sin y) = -y \sec^2 x$$

$$\frac{dy}{dx} = \frac{-y \sec^2 x}{\tan x + \sin y}$$

✓

$$-\frac{\pi}{2} = \frac{\cancel{\left(\frac{\pi}{2}\right)^2} \frac{1}{\cos^2(0)} + \cos\frac{\pi}{2}^{0}}{-1 \ (\cancel{\tfrac{\pi}{2}})} =$$

(b) Find the equation of the tangent to the curve at $\left(0, \frac{\pi}{2}\right)$.

$$\text{slope} = \frac{dy}{dx} = \frac{-\left(\frac{\pi}{2}\right)\frac{1}{\cos^2(0)}}{\tan 0 + \sin\left(\frac{\pi}{2}\right)} = \frac{-\frac{\pi}{2} \cdot \frac{1}{1}}{0 + 1} = -\frac{\pi}{2}$$

$y = mx + b$

$y = -\frac{\pi}{2}x + b$

✓

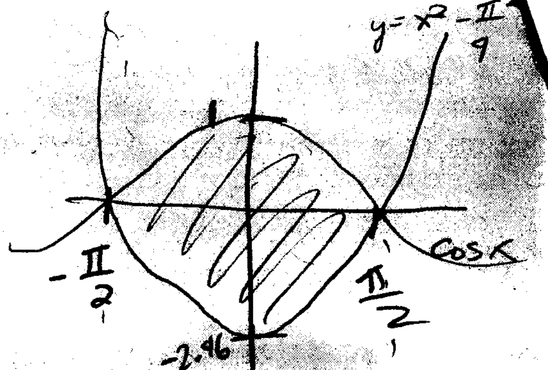
$\frac{\pi}{2} = -\frac{\pi}{2}(0) + b$

$\frac{\pi}{2} = b$

So tangent is

$$\boxed{y = -\frac{\pi}{2}x + \frac{\pi}{2}}$$

5. (7 points) Find the area between $y = \cos x$ and $y = x^2 - \frac{\pi^2}{4}$.



$0 = \cos x$

$= \pm\frac{\pi}{2}$

roots

$y = x^2 - \frac{\pi^2}{4}$

$= \left(x - \frac{\pi}{2}\right)\left(x + \frac{\pi}{2}\right)$

roots $= \pm\frac{\pi}{2}$

Area bounded by $\cos x$ on top and $x^2 - \frac{\pi^2}{4}$ bottom:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overset{f(x)}{\cos x}\, dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overset{g(x)}{x^2 - \frac{\pi^2}{4}}\, dx$$

$$\overset{F}{\left[\sin x\right]_{-\frac{\pi}{2}\,a}^{\frac{\pi}{2}\,b}} - \overset{G}{\left[\frac{x^3}{3} - \frac{\pi^2}{4}x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}$$

$$\left[\sin\frac{\pi}{2} - \sin -\frac{\pi}{2}\right] - \left[\frac{1}{3}\left(\frac{\pi}{2}\right)^3 - \frac{\pi^2}{4}\left(\frac{\pi}{2}\right) + \frac{1}{3}\left(+\frac{\pi}{2}\right)^3 + \frac{\pi^2}{4}\left(-\frac{\pi}{2}\right)\right]$$

$$\frac{1}{3}\left(\frac{\pi}{2}\right)^3 - \left(\frac{\pi}{2}\right)^3 + \frac{1}{3}\left(\frac{\pi}{2}\right)^3 - \left(\frac{\pi}{2}\right)^3$$

$$[1 + +1] \qquad [2.58 - 7.75]$$

$$2 - $$

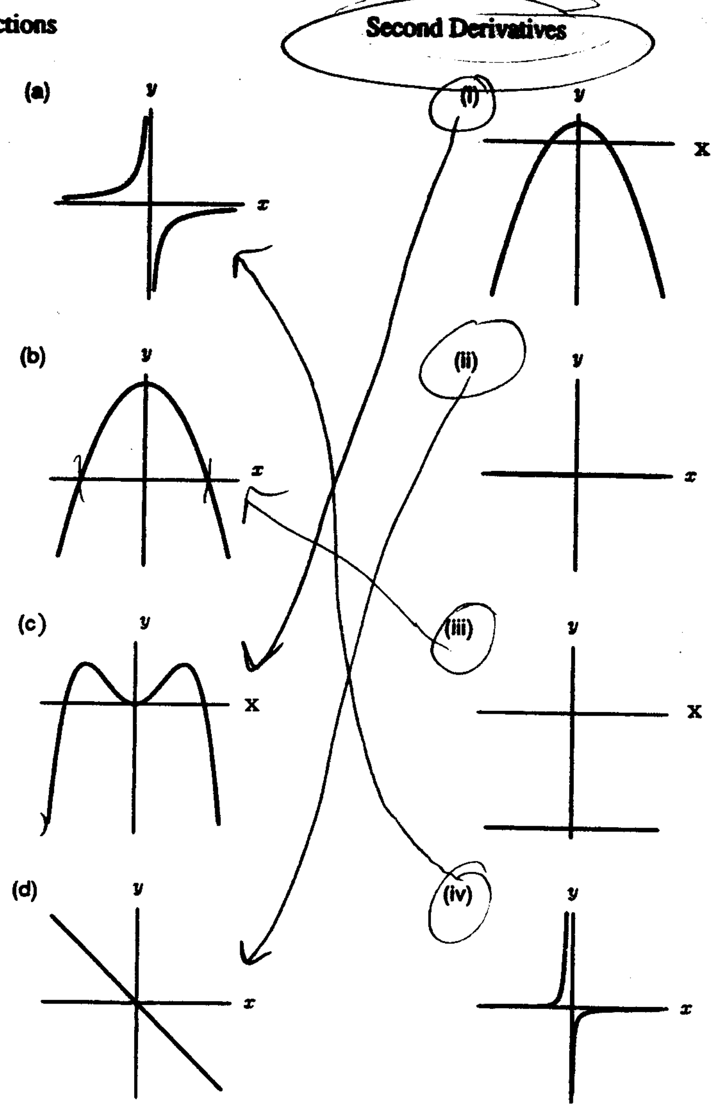
$$2 - [-5.17]$$

$$2 + [+5.17]$$

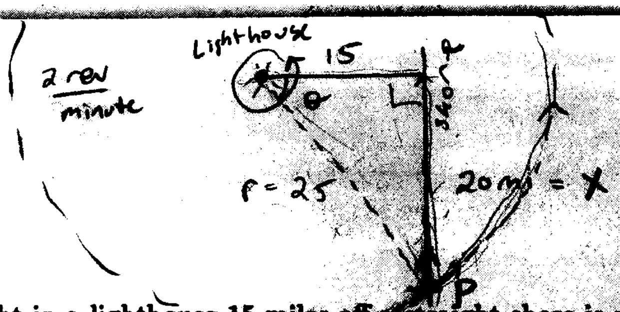
$$2 + 5.17 = \boxed{7.17}$$

6. (8 points) Each graph in the right-hand column represents the **second derivative** of some function in the left-hand column. Match the functions and their second derivatives.

determines concavity



Function (a) has second derivative __iv__.
Function (b) has second derivative __iii__.
Function (c) has second derivative __i__.
Function (d) has second derivative __ii__.



7. **(9 points)** A searchlight in a lighthouse 15 miles off a straight shore is revolving at the rate of 2 revolutions/minute. At what speed does the beam of light pass a point 20 miles down the shore from the lighthouse?

~~X = 25~~ Perimeter at r = 25

ignore this

Question unclear; according to proctor I asked he said you wanted the speed along perimeter (r=25):

$P = 2\pi r$

$P = 2\pi(25)$

$P = 50\pi$

If 2 revolutions per minute:

$$\frac{2 \cdot P}{1\,\text{min}} = 2\,\frac{(50\pi)}{1\,\text{min}} = 100\pi \text{ miles/minute}$$

≈ 314 miles/minute

If you want speed of beam as it passes through a point **25** miles from lighthouse along perimeter of circle (r=25) then speed is 100π miles/minute.

However, I suspect you want the speed as it passes the point going up the shoreline! (Involves calculus) So here is:

$$\frac{d\theta}{dt} = \frac{2\pi \text{ rads}}{2 \text{ min}} = \pi$$

$$\tan\theta = \frac{x}{15}$$

$$15\tan\theta = x$$

$$\frac{dx}{dt} = 15 \cdot \sec^2\theta \cdot \left(\frac{d\theta}{dt}\right) \quad -1$$

$$\frac{dx}{dt} = 15\,\frac{1}{\cos^2\left(\frac{15}{25}\right)}\,(\pi)$$

$$= 15\,(1.468)\,\pi \quad {}^{-1}$$

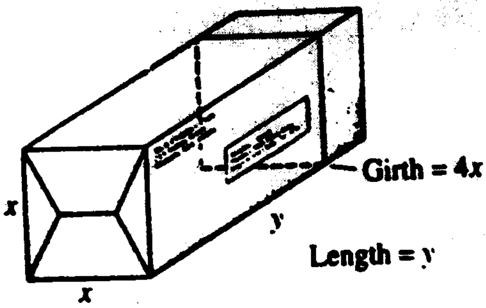
$$\boxed{\frac{dx}{dt} = 69.2 \text{ miles a minute}}$$

(speed of beam as it goes along shore thru pt. P)

(I think this is interpretation you wanted: if x is distance up shore from point P;

$$\boxed{\cancel{\frac{dx}{dt}} = 69.2 \text{ miles/min}}$$

8. (9 points) The U. S. Postal Service will accept a box for domestic equipment only if the sum of its length and girth (distance around) does not exceed 108 inches. Find the dimensions of the largest acceptable box with a square end. (Greatest volume!) Max



$4x + y \le 108$

$y \le 108 - 4x$

Volume $= x^2 \cdot y$

Volume $= x^2 \cdot y$

Volume $= x^2 \cdot (108 - 4x) = 108x^2 - 4x^3$

$V' = 2 \cdot 108x - 12x^2$

$0 = 216x - 12x^2$

$0 = x(216 - 12x)$

$= 216 = 12x$

$18_{in.} = x$

$4(18) + y \le 108$

$72 + y \le 108$

$y \le 36$

Dimensions

$\boxed{18 \times 18 \times 36 \text{ inches}}$

$= 11{,}664 \text{ in}^3$

pt. To left:

$17 \times 17 \times 40 = 11560 < 11{,}664$

pt. To Rt:

$19 \times 19 \times 32 = 11552 < 11{,}664$

so Max.

$V'' = 216 - 24x$

$= 216 - 24(18)$

$= 216 - 432$

$V'' = -216$ negative it's max!

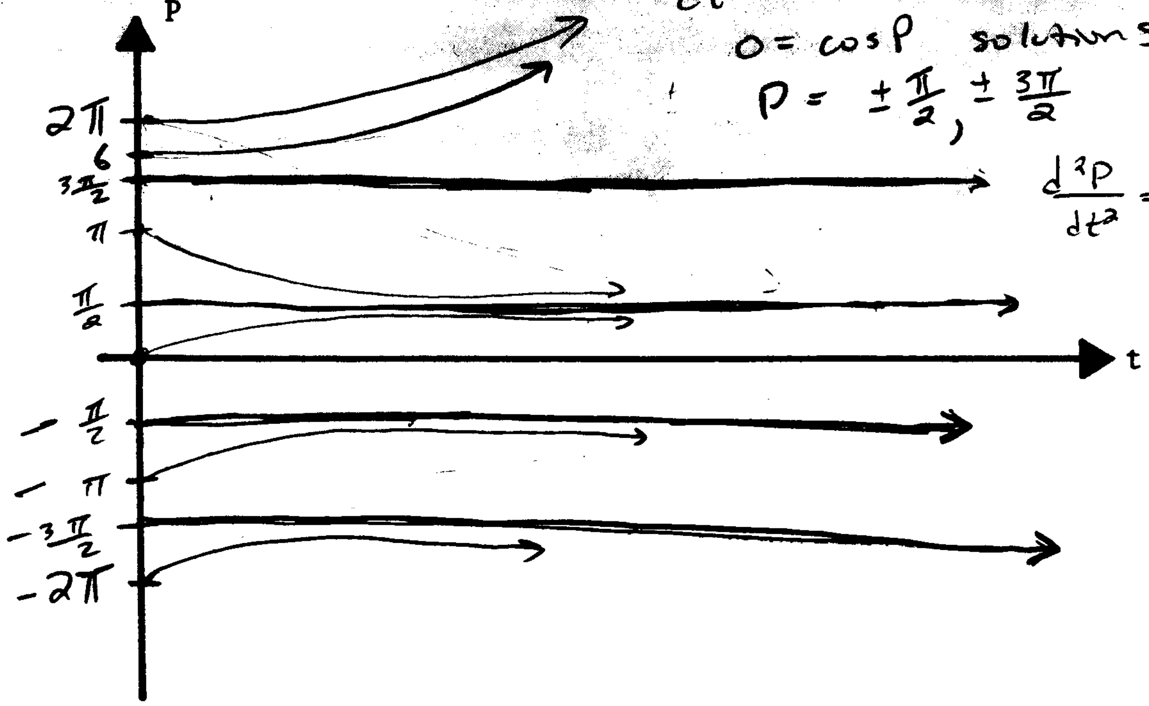
(concave down)

9. **(10 points) (a) Sketch a representative family of solution curves for the differential equation $\frac{dP}{dt} = \cos P$ for $-2\pi \leq P \leq 2\pi$ on the axis below.**

$\frac{dP}{dt} = \cos P$

$0 = \cos P$ solutions for $P = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

$\frac{d^2P}{dt^2} = -\sin P$



(b) What happens to ~~$P(t)$ as $t \to \infty$ if $P(0) = 6$?~~

$P(t)$ approaches ∞ (gets larger) if starting at $P(0) = 6$. ($\frac{dP}{dt} = \cos(6)$ is $.96$ positive.

-2

(c) What happens to $P(t)$ as $t \to \infty$ if $P(0) = \frac{3\pi}{2}$?

$P(0) = \frac{3\pi}{2}$ stays constant as $t \to \infty$.

(since $\frac{dP}{dt} = 0$ at $\cos(\frac{3\pi}{2})$)

✓

10. (10 points) Eugene Saperstein is driving south on the Panamaniac Highway in his "Volga" sedan. Suddenly, having seen a roadblock, he steps on the brakes and decelerates at $-20m/s^2$. If it takes him 90m to come to a stop, how fast was he driving (in m/s) when he stepped on the brakes?

$a(t) = -20 m/s^2$

$V(t) = -20t + V_0$ Let $s_0 = 0$

$s(t) = -\frac{20t^2}{2} + V_0 t + s_0$

$s(t) = -10t^2 + V_0 t + s_0$

How many seconds before he stopped (Set $V(t) = 0$)

$0 = -20t + V_0$

$V_0 = 20t$

Substitute into

$s(t) = -10t^2 + V_0 t$

$90 = -10t^2 + V_0 t$

$90 = -10t^2 + (20t)t$

$90 = -10t^2 + 20t^2$

$90 = 10t^2$

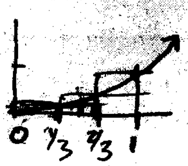
$\sqrt{\frac{90}{10}} = t^2$

$\pm 3 = t$ 3 seconds before stopping he applied brakes.

so $V_0 = 20(3)$ C

$\boxed{V_0 = 60 \ m/s}$

$\Delta x = \frac{b-a}{n}$



11. (8 points) (a) How many subdivisions would be needed to approximate $\int_0^1 \frac{(1+x^2)^5}{100}\,dx$ with an error of no more than 0.11?

$\Delta x (U - L) < .11$

$\Delta x = \frac{1-0}{n}$

$\Delta x = \frac{1}{n}$

$\frac{1}{n}\left(\frac{(1+1^2)^5}{100} - \frac{(1+0^2)^5}{100}\right) < .11$

$\frac{1}{n}\left(\frac{(2)^5}{100} - \frac{1}{100}\right) < .11$

$\boxed{2.81 = \frac{(.32 - .01)}{.11} < n}$

$\therefore$ n must be at least 3

(b) Using the number of subdivisions n you found in part (a), find an upper bound U for $\int_0^1 \frac{(1+x^2)^5}{100}\,dx$.

RHS $\quad \Delta x = \frac{1}{3}$

$\frac{1}{3}\left(\left(\frac{(1+\frac{1}{3}^2)^5}{100}\right) + \frac{(1+\frac{2}{3}^2)^5}{100} + \frac{(1+1^2)^5}{100}\right) =$

$\frac{1}{3}(.129 + .0629 + .32)$

$\frac{1}{3}(.5119)$

$U \approx .17$

(c) Using the number of subdivisions n you found in part (a), find a lower bound L for $\int_0^1 \frac{(1+x^2)^5}{100}\,dx$

LHS

$= \frac{1}{3}\left(\left(\frac{(1+0^2)^5}{100}\right) + \left(\frac{(1+\frac{1}{3}^2)^5}{100} + \frac{(1+\frac{2}{3}^2)^5}{100}\right)\right)$

$= \frac{1}{3}(.01 + .129 + .0629)$

$= \frac{1}{3}(.2019)$

$L \approx .0673 \approx .07$

Check:
$U - L =$
$.17 - .07 = .10 < .11$ ✓

12. **(13 points) Consider the function** $f(x) = \frac{\ln x}{x}$. $\frac{f(x)}{g(x)}$

$$\frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}$$

$1 - \ln x > 0$

$1 > \ln x$

(a) Show that $f'(x) = \frac{1 - \ln x}{x^2}$.

$$f'(x) = \frac{\frac{1}{\cancel{x}}(\cancel{x}) - \ln x\,(1)}{x^2}$$

$$\boxed{f'(x) = \frac{1 - \ln x}{x^2}}$$

(b) Show that $f''(x) = \frac{2\ln x - 3}{x^3}$.

$$f''(x) = \frac{-\frac{1}{\cancel{x}}(\cancel{x}^2) - 2x(1 - \ln x)}{(x^2)^2}$$

$$f''(x) = \frac{-x - 2x + 2x\ln x}{x^4}$$

$$= \frac{\cancel{x}(2\ln x - 3)}{\cancel{x^4}\; x^3} = \boxed{\frac{2\ln x - 3}{x^3}}$$

(c) In what intervals is f **increasing?** where $f'(x)$ is positive

where $\ln x < 1$, $x > 0$

$0 < x < e$

(d) In what intervals is f **concave up?** where $f''(x)$ is positive

$2\ln x - 3 > 0$ $\frac{2\ln x}{2} > \frac{3}{2}$ $\boxed{x > e^{3/2}}$

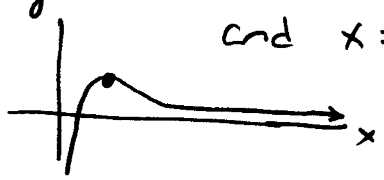
(e) Provide the coordinates of all local max and local min, if any, and indicate what they are. Do not approximate.

$f(x) = \frac{\ln x}{x} = \frac{\ln e}{e} = \frac{1}{e}$

Continued on next page.

No local min (approaches $y=0$ and $x=0$)

local max: $\left(e, \frac{1}{e}\right)$



12, continued.

(f) Provide the coordinate of all inflection points. Do not approximate.

Inflection pt at $\left(e^{3/2}, \frac{1.5}{e^{3/2}}\right)$ $\qquad \frac{\ln e^{3/2}}{e^{3/2}} = \frac{1.5}{e^{3/2}}$

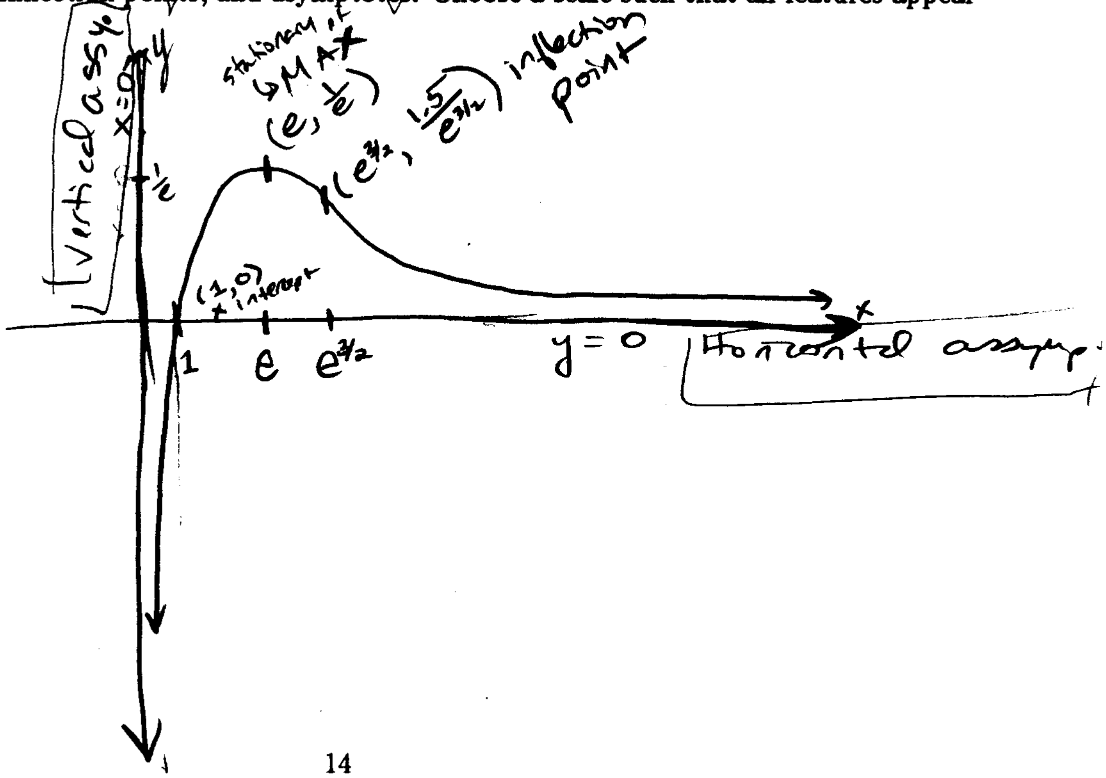
(g) Does this function have any vertical asymptotes? Explain.

Yes, at $x = 0$ (Function $\frac{\ln x}{x}$ is undefined at $x=0$ both in numerator + denominator

(h) Does this function have any horizontal asymptotes?

Yes at $y = 0$.

(i) Sketch the graph of the above function, labeling (without approximating) all intercepts, stationary points, inflection points, and asymptotes. Choose a scale such that all features appear clear.



$f(x) = \frac{\ln x}{x}$

$0 = \frac{\ln x}{x}$

$0 = \ln x$

$1 = x$