Name of Student: ....

(Your Instructor's Name: ....)

First Mid-Term of Math 1a

October 17, 2000 (Tuesday)
7 p.m. - 9 p.m., Science Center Hall C & E

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Question	Points	Score
1	13	
2	13	
3	13	
4	12	
5	13	
6	12	
7	12	
8	12	
Total	100	

- *You have TWO hours to complete this examination.*
- *No calculators are allowed.*
- *No partial credit can be given for unsubstantiated answers.*
- *Use the back of the page if more space is needed for your answer (with an indication that your answer is continued on the back of the page).*

1. Let $f(x) = |x+2| - 2|x| + |x-3|$.

   (a) Evaluate $f(-5)$, $f(\frac{1}{2})$, and $f(3)$.

   (b) Sketch the graph of $f(x)$.

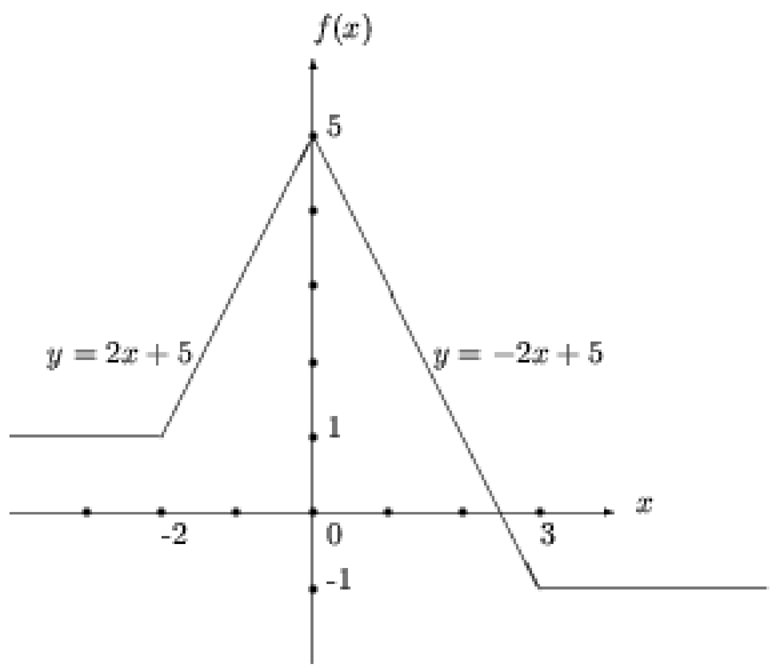
   (c) Let $g(x) = \frac{1}{x}$ and $h(x) = \sqrt{x-3}$. Find the natural domains of $g \circ f$ and $h \circ f$. (Hint: refer to your graph!)

Solution:

(a)

$$f(-5) = |-5+2| - 2|-5| + |-5-3| = 3 - 2 \times 5 + 8 = 1.$$
$$f\left(\frac{1}{2}\right) = \left|\frac{1}{2}+2\right| - 2\left|\frac{1}{2}\right| + \left|\frac{1}{2} - 3\right| = \frac{5}{2} - 1 + \frac{5}{2} = 4.$$
$$f(3) = |3+2| - 2|3| + |3-3| = 5 - 6 = -1.$$

(b) The graph of $y = f(x)$ is as follows.



The computation for the graph is as follows.

$f(x) = -(x+2) + 2x - (x-3) = 1 \quad$ for $x \le -2$.
$f(x) = (x+2) + 2x - (x-3) = 2x + 5 \quad$ for $-2 \le x \le 0$.
$f(x) = (x+2) - 2x - (x-3) = -2x + 5 \quad$ for $0 \le x \le 3$.
$f(x) = (x+2) - 2x + (x-3) = -1 \quad$ for $3 \le x$.

(c) The natural domain of $g \circ f$ is defined by $f(x) \neq 0$, which according to the graph is the set of all values of x except $x = \frac{5}{2}$ which is computed from $-2x + 5 = 0$.

The natural domain of $h \circ f$ is defined by $f(x) \ge 3$, which according to the graph is the interval $[-1, 1]$. The left-end point of the interval $[-1, 1]$ is obtained by solving $2x + 5 = 3$ and the right-end point of the interval $[-1, 1]$ is obtained by solving $-2x + 5 = 3$.

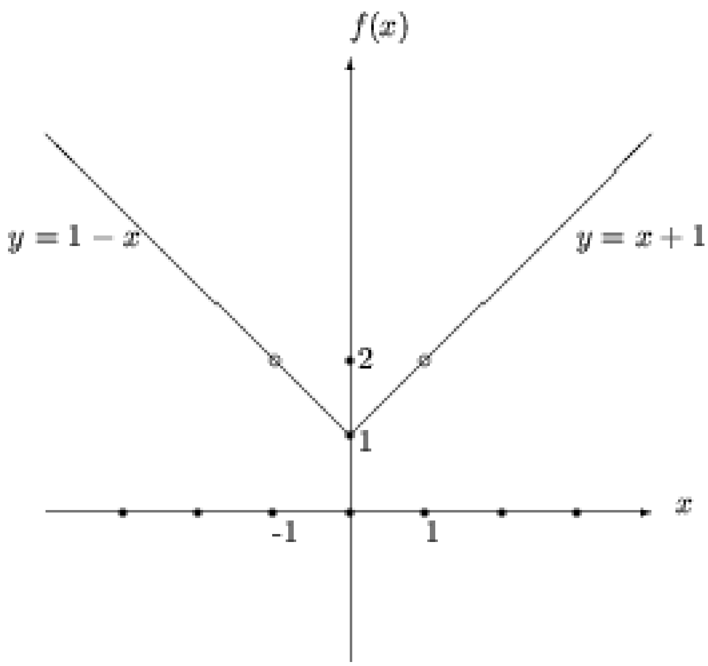
2. Sketch the graph of

$$f(x) = \frac{x^2 - 1}{|x| - 1}.$$

Where is f continuous? Are there any removable discontinuities? (*Recall:* a discontinuity is a removable discontinuity if the limit of the function exists at that point.)

I. $x \leq 0$. $f(x) = \frac{x^2-1}{-(x+1)}$ for $x \neq -1$.

II. $x \geq 0$. $f(x) = \frac{x^2-1}{x-1}$ for $x \neq 1$.



f is continuous for $x \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

f has removable discontinuity at $x = -1$ and $x = 1$, because

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1}(x + 1) = 2$$

exists and because

$$\lim_{x \to -1} \frac{x^2 - 1}{-(x+1)} = \lim_{x \to -1} (1 - x) = 2$$

exists, but f is undefined for $x = -1$ and $x = 1$.

3. Let $f(x) = \frac{3-x}{x^2-2x-8}$. Evaluate the following limits.

(a) $\lim_{x\to\infty} f(x)$. (b) $\lim_{x\to-\infty} f(x)$. (c) $\lim_{x\to 1^+} f(x)$.

(d) $\lim_{x\to 1^-} f(x)$. (e) $\lim_{x\to 4^+} f(x)$. (f) $\lim_{x\to 4^-} f(x)$.

Solution:

(a)

$$\lim_{x\to\infty} \frac{3-x}{x^2-2x-8} = \lim_{x\to\infty} \frac{\frac{3}{x^2}-\frac{1}{x}}{1-\frac{2}{x}-\frac{8}{x^2}} = \frac{\lim_{x\to\infty}\left(\frac{3}{x^2}-\frac{1}{x}\right)}{\lim_{x\to\infty}\left(1-\frac{2}{x}-\frac{8}{x^2}\right)} = \frac{0}{1} = 0.$$

(b)

$$\lim_{x\to-\infty} \frac{3-x}{x^2-2x-8} = \lim_{x\to-\infty} \frac{\frac{3}{x^2}-\frac{1}{x}}{1-\frac{2}{x}-\frac{8}{x^2}} = \frac{\lim_{x\to-\infty}\left(\frac{3}{x^2}-\frac{1}{x}\right)}{\lim_{x\to-\infty}\left(1-\frac{2}{x}-\frac{8}{x^2}\right)} = \frac{0}{1} = 0.$$

(c)

$$\lim_{x\to 1^+} \frac{3-x}{x^2-2x-8} = \frac{\lim_{x\to 1^+}(3-x)}{\lim_{x\to 1^+}(x^2-2x-8)} = \frac{3-1}{1-2-8} = -\frac{2}{9}.$$

(d)

$$\lim_{x\to 1^-} \frac{3-x}{x^2-2x-8} = \frac{\lim_{x\to 1^-}(3-x)}{\lim_{x\to 1^-}(x^2-2x-8)} = \frac{3-1}{1-2-8} = -\frac{2}{9}.$$

(e)

$$\lim_{x\to 4^+} \frac{3-x}{x^2-2x-8} = \lim_{x\to 4^+} \frac{3-x}{(x-4)(x+2)} = \left(\lim_{x\to 4^+}\frac{1}{x-4}\right)\left(\lim_{x\to 4^+}\frac{3-x}{x+2}\right)$$
$$= \left(\lim_{x\to 4^+}\frac{1}{x-4}\right)\left(\frac{-1}{6}\right) = -\infty.$$

(f)

$$\lim_{x\to 4^-} \frac{3-x}{x^2-2x-8} = \lim_{x\to 4^-} \frac{3-x}{(x-4)(x+2)} = \left(\lim_{x\to 4^-}\frac{1}{x-4}\right)\left(\lim_{x\to 4^-}\frac{3-x}{x+2}\right)$$
$$= \left(\lim_{x\to 4^-}\frac{1}{x-4}\right)\left(\frac{-1}{6}\right) = \infty.$$

4. Compute the derivative of $f(x) = \frac{x^3-1}{x^2+x}$.

Solution:

$$
\begin{aligned}
f'(x) &= \frac{3x^2\left(x^2+x\right)-(2x+1)\left(x^3-1\right)}{\left(x^2+x\right)^2} \\
&= \frac{3x^4+3x^3-2x^4-x^3+2x+1}{\left(x^2+x\right)^2} = \frac{x^4+2x^3+2x+1}{x^2(x+1)^2}.
\end{aligned}
$$

5. Compute $f'(x)$ and $f''(x)$ when $f(x) = \sin^2(x^4 + 1)$.

Solution:

$$\begin{aligned} f(x) &= \sin^2(x^4 + 1). \\ f'(x) &= 8x^3 \cos(x^4 + 1) \sin(x^4 + 1) \\ &= 4x^3 \sin(2x^4 + 2). \\ f''(x) &= 24x^2 \sin(x^4 + 1) \cos(x^4 + 1) + 32x^6(\cos^2(x^4 + 1) - \sin^2(x^4 + 1)) \\ &= 12x^2 \sin(2x^4 + 2) + 32x^6 \cos(2x^4 + 2). \end{aligned}$$

6. Find all lines tangent to the graph of $y = x^2$ which pass through the point $(1, -3)$.

Solution 1: Our strategy is as follows: first, we find all lines tangent to the graph of $y = x^2$. Then we determine which of these pass through the point $(1, -3)$.

The tangent line to $y = x^2$ at the point $(x_0, y_0) = (x_0, x_0^2)$ has slope $\frac{dy}{dx}(x_0) = 2x_0$. Therefore, using the point-slope method for determining lines, this tangent line has equation

$$\begin{aligned} y &= (2x_0)(x - x_0) + x_0^2 \\ &= 2x_0 x - x_0^2. \end{aligned}$$

We now have the equation for the tangent line to $y = x^2$ at the point (x_0, x_0^2), and we want to determine for which values of x_0 this passes through the point $(1, -3)$. This requires, simply:

$$-3 = 2x_0(1) - x_0^2,$$

i.e.,

$$x_0^2 - 2x_0 - 3 = 0.$$

Solving this equation (either using the quadratic formula, or by noticing that this polynomial factors as $(x_0 - 3)(x_0 + 1)$), the desired values of x_0 are $x_0 = -1, 3$. Finally, since the question asks for the equations of the lines, we substitute $x_0 = -1, 3$ back into our general equation for the tangent line through (x_0, x_0^2) to get

$$\begin{aligned} y &= -2x - 1 \quad \text{and} \\ y &= 6x - 9\,. \end{aligned}$$

Solution 2: This time, our strategy is to write down all lines through $(1, -3)$, and determine which of these are tangent to the curve $y = x^2$. By the point-slope method, all lines through $(1, -3)$ are of the form

$$y = m(x - 1) + 3$$

except the vertical line $x = 1$. Which of these are tangent to $y = x^2$? Certainly the vertical line $x = 1$ isn't, since no tangent lines to $y = x^2$ are vertical, so we only need to think about the lines $y = m(x - 1) + 3$.

Suppose $y = m(x-1)+3$ is tangent to $y = x^2$ at the point $(x_0, y_0) = (x_0, x_0^2)$. Since the tangent line to $y = x^2$ at that point has to have slope $\frac{dy}{dx}(x_0) = 2x_0$, we obtain the condition

$$x_0^2 = (2x_0)(x_0 - 1) + 3.$$

(Alternately, one could get this by saying: the slope from $(1, -3)$ to (x_0, x_0^2) is $\frac{x_0^2+3}{x_0-1}$, and this must equal $2x_0$.) This equation can again be solved to yield $x_0 = -1$, 3, and therefore the same two lines as in Solution 1.

Comments: The most common error was to interpret the point $(1, -3)$ as the point of tangency (which doesn't make sense, since it isn't even on the graph!) and to conclude that since the tangent line must have slope $2 \cdot 1 = 2$ and pass through $(1, -3)$, it must be the line $y = 2x - 5$. This was worth 1 point for correctly differentiating $y = x^2$.

The second most common error was to write down the collection of lines $y = m(x-1) - 3$ through $(-1, 3)$, then to assert that since the slope is $m = 2x$, the tangent line must be $y = (2x)(x-1) - 3 = 2x^2 - 2x - 3$. (This is very much in the right direction: if you realize that what we have written down, above, is the *condition* for one of these line to be tangent to the curve at (x, y), then you substitute $x^2 = y = 2x^2 - 2x - 3$ and happily solve for $x = -1$, 3.)

The source of confusion, here, is that x has been used to denote two different things: the variable in the equation for the line, and the coordinate of the point of tangency. If one is careful to think through what one is doing, e.g. by saying "suppose the line $y = m(x-1) - 3$ is tangent to the curve $y = x^2$ at the point $(x_0, y_0) = (x_0, x_0^2)$", it's impossible to make the above mistake.

This was worth 5 points, since somewhat more of the correct working is present; however, if you seemed to be claiming that the equation $y = 2x^2 - 2x - 3$ was the equation of a line, points were deducted for claiming something that you should know is nonsense.

7. Suppose $g(1) = 4$, $g'(1) = 3$, and $g''(1) = -2$. Suppose also that $f(4) = 6$, $f'(4) = -1$, and $f''(4) = 5$. What are the values of the first and second derivatives of $(f \circ g)(x)$ at $x = 1$?

Solution:

First, note the equivalence of the expressions $f(g(x))$ and $(f \circ g)(x)$. We will use both notations interchangably below. Recall from the chain rule that

$$(f \circ g)'(x) = f'(g(x))\, g'(x). \qquad (*)$$

Thus, when $x = 1$ we have

$$(f \circ g)'(1) = f'(g(1))\, g'(1) = f'(4)\, g'(1) = -1 \cdot 3 = -3.$$

To answer the second part, we can us the product rule to differentiate the equation $(*)$ and obtain

$$(f \circ g)''(x) = \Big((f' \circ g)(x)\, g'(x)\Big)' = (f' \circ g)'(x)\, g'(x) + (f' \circ g)(x)\, g''(x).$$

We again apply $(*)$ to see that

$$(f' \circ g)'(x) = f''(g(x))\, g'(x).$$

After simplifying and plugging in $x = 1$, we see that

$$(f \circ g)''(1) = f''(g(1))\, (g'(1))^2 + (f' \circ g)(1)\, g''(1) = 5 \cdot 3^2 + (-1)(-2) = 47.$$

8. Let

$$f(x) = \begin{cases} x^n \sin\left(\frac{1}{x^2}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$$

(a) Find the smallest integer value for n such that $f(x)$ is continuous at $x = 0$.

(b) Find the smallest integer value for n such that $f(x)$ is differentiable at $x = 0$.

(c) Find the smallest integer value for n such that $f''(x)$ exists at $x = 0$.

Solution:

(a) Since $\sin\left(\frac{1}{x^2}\right)$ assumes the value 1 at $x = \sqrt{\frac{1}{(2k+\frac{1}{2})\pi}}$ and assumes the value 0 at $x = \sqrt{\frac{1}{(2k+1)\pi}}$ for any integer k, in order for

$$\lim_{x\to 0} x^n \sin\left(\frac{1}{x^2}\right) = 0$$

to hold, it is necessary and sufficient for the integer n to satisfy $n \geq 1$. Thus the smallest integer value for n to make $f(x)$ continuous at $x = 0$ is $n = 1$.

(b) For $f'(0)$ to exist, the function $f(x)$ must first be continuous at $x = 0$ and n must be at least 1 to start with. In order for the difference quotient

$$\frac{f(x) - f(0)}{x - 0} = x^{n-1} \sin\left(\frac{1}{x^2}\right)$$

to approach a finite limit as $x \to 0$, it is necessary and sufficient for the integer n to satisfy $n - 1 \geq 1$. Thus the smallest integer value for n to make $f(x)$ differentiable at $x = 0$ is $n = 2$. In this case, $f'(0) = 0$.

(c) To consider the existence of $f''(0)$, we must first have the existence of $f'(0)$, which implies that n must be at least 2 to start with.

$$f'(x) = nx^{n-1} \sin\left(\frac{1}{x^2}\right) + 2x^{n-3} \cos\left(\frac{1}{x^2}\right).$$

For $f''(0)$ to exist, the function $f'(x)$ must first be continuous at $x = 0$. It means that the limit of $2x^{n-3}\cos\left(\frac{1}{x^2}\right)$ must be 0 as $x \to 0$, which implies that $n \geq 4$. For the computation of $f''(0)$ the difference quotient is

$$\frac{f'(x) - f'(0)}{x - 0} = nx^{n-2}\sin\left(\frac{1}{x^2}\right) + 2x^{n-4}\cos\left(\frac{1}{x^2}\right).$$

For $n \geq 4$, the difference quotient has a finite limit as $x \to 0$ if and only if $n \geq 5$. Thus the smallest integer value of n for $f''(0)$ to exist at $x = 0$ is $n = 5$.