Section 3.7

“Derivatives of logarithmic functions”

Rules of exponentials and logarithms

1. $a^{b+c} = a^b a^c$
2. $a^{b-c} = \frac{a^b}{a^c}$
3. $(a^b)^c = a^{bc}$
4. $a^{-1} = \frac{1}{a}$
5. $a^0 = 1.$

$$\boxed{a^{\log_a b} = b}$$

1. $\log_a(bc) = \log_a(b) + \log_a(c)$
2. $\log_a \frac{b}{c} = \log_a(b) - \log_a(c)$
3. $\log_a(b^c) = c \log_a b$
4. $\log_a \frac{1}{b} = -\log_a b$
5. $\log_a 1 = 0.$

Derivatives of logarithmic functions

Theorem. *1.*

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

2. If $a > 0$,

$$\frac{d}{dx} \log_a x = \frac{1}{(\ln a)x}.$$

Derivatives of exponential functions

Theorem. *If a is any positive number,*

$$\frac{d}{dx}a^x = (\ln a)a^x.$$

Tips for Logarithmic Differentiation

1. Start with $y = f(x)$.

2. Take the natural logarithm of both sides.

3. Use properties of logarithms to simplify the right-hand side.

4. Take the derivative. On the left you will have
$$\frac{d}{dx} \ln y = \frac{1}{y}\frac{dy}{dx}.$$

5. Multiply both sides by y and substitute $y = f(x)$.

Theorem.

$$\lim_{h \to 0} \ln (1 + h)^{1/h} = 1.$$

Proof. Let $f(x) = \ln x$. We know $f'(1) = 0$. This means

$$\begin{aligned} 1 &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{\ln(1+h) - \ln 1}{h} \\ &= \lim_{h \to 0} \ln(1+h)^{1/h}. \end{aligned}$$

□

Theorem.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Proof. First, exponentiate the last theorem:

$$e^{\lim_{h\to 0}\ln(1+h)^{1/h}} = e^1$$

$$\lim_{h\to 0} e^{\ln(1+h)^{1/h}} = e^1$$

$$\lim_{h\to 0}(1+h)^{1/h} = e.$$

Now, if n is any positive number, let $h=\frac{1}{n}$. Then as $n\to\infty$, $h\to 0$, and so

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{h\to 0}(1+h)^{1/h} = e.$$

□

Questions

1. **[Q] True** or **False**. $\frac{d}{dx}\ln(\pi) = \frac{1}{\pi}$.

2. **[Q]** Your calculus book says that $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$. This means:

(a) e is not really a number because it is a limit

(b) e cannot be computed

(c) the sequence of numbers $\left(\frac{2}{1}\right), \left(\frac{3}{2}\right)^2, \left(\frac{4}{3}\right)^3, \ldots, \left(\frac{101}{100}\right)^{100}, \ldots$ get as close as you want to the number e

3. **[P]** When you read in the newspaper thing like inflation rate, interest rate, birth rate, etc., it always means $\frac{f'}{f}$, not f' itself.

   True or **False**. $\frac{f'}{f}$ is not the derivative of a function.