Mathematics 1a, Section 3.3 Solutions

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1. a.

$$s = f(t) = t^3 - 12t^2 + 36t$$
$$v(t) = f'(t) = 3t^2 - 24t + 36$$

b.

$$v(3) = 27 - 72 + 36 = -9\text{m/s}$$

c. The particle is at rest when $v(t) = 0$.

$$3t^2 - 24t + 36 = 0$$
$$3(t-2)(t-6) = 0$$
$$t = \{2, 6\}$$

d. The particle is moving in the positive direction when $v(t) > 0$.

$$3(t-2)(t-6) > 0$$
$$0 \leq t < 2 \text{or} t > 6$$

e. Since the particle is moving forward and backward, we need to calculate the distance traveled in the intervals $[0, 2]$, $[2, 6]$ and $[6, 8]$ separately.

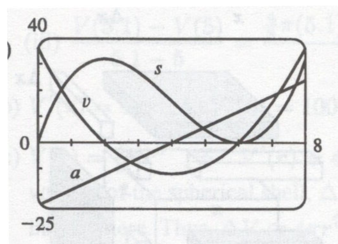
$$|f(2) - f(0)| = |32 - 0| = 32$$
$$|f(6) - f(2)| = |0 - 32| = 32$$
$$|f(8) - f(6)| = |32 - 0| = 32$$

So the total distance traveled is $32 + 32 + 32 = 96\text{m}$.

f. **g.**

$$a(t) = v'(t) = 6t - 24$$
$$a(3) = 6(3) - 42 = -6\text{m/s}^2$$

h.



i. The particle is speeding up when v and a have the same sign. This occurs when $2 < t < 4$ and when $t > 6$. It is slowing down when v and a have opposite signs; that is, when $0 \leq t < 2$ and when $4 < t < 6$.

14. a.

$$F = \frac{GmM}{r^2} = (GmM)r^{-2}$$
$$\frac{dF}{dr} = -2(GmM)r^{-3} = -\frac{2GmM}{r^3}$$

which is the rate of change of the force with respect to the distance between the bodies. The minus sign indicates that as the distance r between the bodies increases, the magnitude of the force F exerted by the body of mass m on the body of mass M is decreasing.

b. Given $F'(20,000) = -2$, find $F'(10,000)$.

$$-2 = -\frac{2GmM}{20,000^3}$$
$$GmM = 20,000^3$$
$$F'(10,000) = -\frac{2(20,000^3)}{10,000^3} = -2 \cdot 2^3 = -16\text{N/km}$$

20. a. After an hour the population is $n(1) = 3 \cdot 500$; after two hours it is $n(2) = 3(3 \cdot 500) = 3^2 \cdot 500$; after three hours, $n(3) = 3(3^2 \cdot 500) = 3^3 \cdot 500$; after four hours $n(4) = 3^4 \cdot 500$.

From this pattern, we see that the population after t hours is $n(t) = 3^t \cdot 500 = 500 \cdot 3^t$.

b. From 5 in Section 3.1, we have

$$\frac{d}{dx}(3^x) \approx (1.10)3^x$$

Thus, for $n(t) = 500 \cdot 3^t$,

$$\frac{dn}{dt} = 500\frac{d}{dt}(3^t) \approx 500(1.10)3^t$$
$$\frac{dn}{dt}|_{t=6} \approx 500(1.10)3^6 \approx 400{,}950\text{bacteria/hour}$$

24. a.

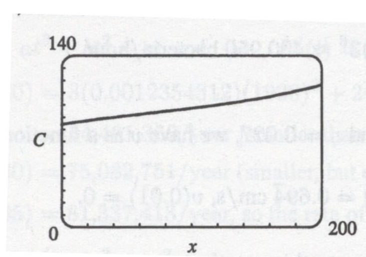
$$C(x) = 84 + 0.16x - 0.0006x^2 + 0.000003x^3$$
$$C'(x) = 0.16 - 0.0012x + 0.00000x^2$$
$$C'(100) = 0.13$$

This is the rate at which the cost is increasing as the 100th item is produced.

b.

$$C(101) - C(100) = 97.13030299 \approx \$0.13$$

c.



From the graph, we can estimate the x-coordinate of the point of inflection to be between 60 and 80.

d. $C''(x) = -0.0012 + 0.000018x = 0$, so $x = 66\frac{2}{3}$ and $C''(x)$ changes from negative to positive at this value of x. This is where the *marginal cost* changes from decreasing to increasing and so has its minimum value.

28. a. If $\frac{dP}{dt} = 0$, the population is stable (it is constant).

b.

$$\frac{dP}{dt} = 0$$
$$\beta P = r_0 \left(1 - \frac{P}{P_c}\right) P$$
$$\frac{\beta}{r_0} = 1 - \frac{P}{P_c}$$
$$\frac{P}{P_c} = 1 - \frac{\beta}{r_0}$$
$$P = P_c \left(1 - \frac{\beta}{r_0}\right)$$

If $P_c = 10,000$, $r_0 = 5\% = 0.05$, and $\beta = 4\% = 0.04$, then $P = 10,000\left(1 - \frac{4}{5}\right) = 2000$.

c. If $\beta = 0.05$, then $P = 10,000\left(1 - \frac{5}{5}\right) = 0$. There is no stable population.