Math 1a Homework Solutions
Section 4.3 (II)

2. (a) g is concave upward on $(-1, 2)$ and $(7, 8)$.

(b) g is concave downward on $(2, 4)$ and $(4, 7)$.

(c) The only point of inflection is $(2, 2)$. Note that 7 is not in the domain of this function.

4. (a) See the First Derivative Test.

(b) See the Second Derivative Test and the note that precedes Example 5.

6. (a) f is increasing on the intervals where $f'(x) > 0$, namely, $(2, 4)$ and $(6, 9)$.

(b) f has a local maximum where it changes from increasing to decreasing, that is, where f' changes from positive to negative (at $x = 4$). Similarly, where f' changes from negative to positive, f has a local minimum (at $x = 2$ and at $x = 6$).

(c) When f' is increasing, its derivative f'' is positive and hence, f is concave upward. This happens on $(1, 3)$, $(5, 7)$, and $(8, 9)$. Similarly, f is concave downward when f' is decreasing — that is, on $(0, 1)$, $(3, 5)$, and $(7, 8)$.

(d) f has inflection points at $x = 1$, 3, 5, 7, and 8, since the direction of concavity changes at each of these values.

8. (a) $f(x) = 1 + 8x - x^8 \quad \Rightarrow \quad f'(x) = 8 - 8x^7 = 8\left(1 - x^7\right)$. Thus, $f'(x) > 0 \quad \Leftrightarrow \quad 1 - x^7 > 0 \quad \Leftrightarrow$ $x^7 < 1 \quad \Leftrightarrow \quad x < 1$ and $f'(x) < 0 \quad \Leftrightarrow \quad x > 1$. So f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.

(b) f changes from increasing to decreasing at $x = 1$. Thus, $f(1) = 8$ is a local maximum.

(c) $f''(x) = -56x^6$. $f''(x) < 0$ for $x \neq 0$, so f is concave downward on $(-\infty, 0)$ and on $(0, \infty)$ by the Concavity Test. In fact, f is CD on $\mathbb{R}$ because f' is decreasing on $\mathbb{R}$. There are no inflection points.

12. (a) $y = f(x) = x^2 e^x \quad \Rightarrow \quad f'(x) = x^2 e^x + 2xe^x = x(x+2)e^x$. So $f'(x) > 0 \quad \Leftrightarrow \quad x(x+2) > 0 \quad \Leftrightarrow$ either $x < -2$ or $x > 0$. Therefore f is increasing on $(-\infty, -2)$ and $(0, \infty)$, and decreasing on $(-2, 0)$.

(b) f changes from increasing to decreasing at $x = -2$, so $f(-2) = 4e^{-2}$ is a local maximum. f changes from decreasing to increasing at $x = 0$, so $f(0) = 0$ is a local minimum.

(c) $f'(x) = \left(x^2 + 2x\right)e^x \quad \Rightarrow \quad f''(x) = \left(x^2 + 2x\right)e^x + e^x(2x + 2) = e^x\left(x^2 + 4x + 2\right)$. $f''(x) = 0 \quad \Leftrightarrow$ $x^2 + 4x + 2 = 0 \quad \Leftrightarrow \quad x = -2 \pm \sqrt{2}$. $f''(x) < 0 \quad \Leftrightarrow \quad -2 - \sqrt{2} < x < -2 + \sqrt{2}$, so f is concave downward on $\left(-2 - \sqrt{2}, -2 + \sqrt{2}\right)$ and concave upward on $\left(-\infty, -2 - \sqrt{2}\right)$ and $\left(-2 + \sqrt{2}, \infty\right)$. There are inflection points at $\left(-2 - \sqrt{2}, f\left(-2 - \sqrt{2}\right)\right) \approx (-3.41, 0.38)$ and $\left(-2 + \sqrt{2}, f\left(-2 + \sqrt{2}\right)\right) \approx (-0.59, 0.19)$.

14. (a) $y = f(x) = x \ln x \quad \Rightarrow \quad f'(x) = x(1/x) + \ln x = 1 + \ln x$. $f'(x) > 0 \quad \Leftrightarrow \quad \ln x + 1 > 0 \quad \Leftrightarrow$ $\ln x > -1 \quad \Leftrightarrow \quad x > e^{-1}$. Therefore f is increasing on $(1/e, \infty)$ and decreasing on $(0, 1/e)$.

(b) f changes from decreasing to increasing at $x = 1/e$, so $f(1/e) = -1/e$ is a local minimum.

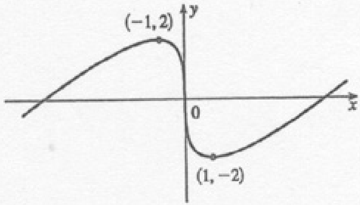
(c) $f''(x) = 1/x > 0$ for $x > 0$. So f is concave upward on its entire domain, and has no inflection point.

20. (a) $Q(x) = x - 3x^{1/3} \Rightarrow Q'(x) = 1 - \dfrac{1}{x^{2/3}} > 0 \Leftrightarrow x^{2/3} > 1 \Leftrightarrow x^2 > 1 \Leftrightarrow x < -1$ or $x > 1$, so Q is increasing on $(-\infty, -1)$, and $(1, \infty)$, and decreasing on $(-1, 1)$.

(b) $Q'(x) = 0 \Leftrightarrow x = \pm 1$; $Q(1) = -2$ is a local minimum, and $Q(-1) = 2$ is a local maximum.

(c) $Q''(x) = \frac{2}{3}x^{-5/3} > 0 \Leftrightarrow x > 0$, so Q is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. Inflection point at $(0, 0)$.

(d)



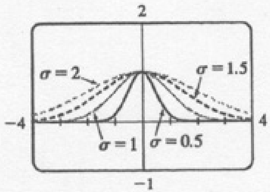
40. (a) As $|x| \to \infty$, $t = -x^2/(2\sigma^2) \to -\infty$, and $e^t \to 0$. The HA is $y = 0$. Since t takes on its maximum value at $x = 0$, so does e^t. Showing this result using derivatives, we have $f(x) = e^{-x^2/(2\sigma^2)} \Rightarrow$ $f'(x) = e^{-x^2/(2\sigma^2)}(-x/\sigma^2)$. $f'(x) = 0 \Leftrightarrow x = 0$. Because f' changes from positive to negative at $x = 0$, $f(0) = 1$ is a local maximum. For inflection points, we find

$$f''(x) = -\frac{1}{\sigma^2}\left[e^{-x^2/(2\sigma^2)} \cdot 1 + xe^{-x^2/(2\sigma^2)}(-x/\sigma^2)\right] = \frac{-1}{\sigma^2}e^{-x^2/(2\sigma^2)}(1 - x^2/\sigma^2).$$

$f''(x) = 0 \Leftrightarrow x^2 = \sigma^2 \Leftrightarrow x = \pm\sigma$. $f''(x) < 0 \Leftrightarrow x^2 < \sigma^2 \Leftrightarrow -\sigma < x < \sigma$. So f is CD on $(-\sigma, \sigma)$ and CU on $(-\infty, -\sigma)$ and (σ, ∞). IP at $\left(\pm\sigma, e^{-1/2}\right)$.

(b) Since we have IP at $x = \pm\sigma$, the inflection points move away from the y-axis as σ increases.

(c)



From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x-axis.