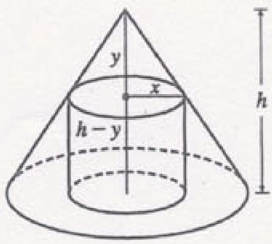
Math 1a Homework Solutions
Section 4.6

2. The two numbers are $x + 100$ and x. Minimize $f(x) = (x + 100)x = x^2 + 100x$. $f'(x) = 2x + 100 = 0 \Rightarrow x = -50$. Since $f''(x) = 2 > 0$, there is an absolute minimum at $x = -50$. The two numbers are 50 and -50.

10. Let b be the length of the base of the box and h the height. The volume is $32{,}000 = b^2 h \Rightarrow h = 32{,}000/b^2$. The surface area of the open box is $b^2 + 4hb = b^2 + 4(32{,}000/b^2)\,b = b^2 + 4(32{,}000)/b$. So $V'(b) = 2b - 4(32{,}000)/b^2 = 2(b^3 - 64{,}000)/b^2 = 0 \Leftrightarrow b = \sqrt[3]{64{,}000} = 40$. This gives an absolute minimum since $V'(b) < 0$ if $0 < b < 40$ and $V'(b) > 0$ if $b > 40$. The box should be $40 \times 40 \times 20$.

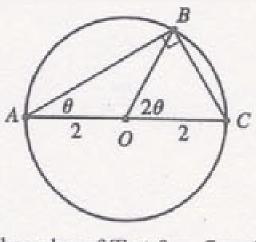
14. The square of the distance from a point (x, y) on the parabola $x = -y^2$ is $x^2 + (y+3)^2 = y^4 + y^2 + 6y + 9 = D(y)$. Now $D'(y) = 4y^3 + 2y + 6 = 2(y+1)(2y^2 - 2y + 3)$. Since $2y^2 - 2y + 3 = 0$ has no real roots, $y = -1$ is the only critical number. Then $x = -(-1)^2 = -1$, so the point is $(-1, -1)$.

20.



By similar triangles, $y/x = h/r$, so $y = hx/r$. The volume of the cylinder is $\pi x^2(h - y) = \pi h x^2 - (\pi h/r)x^3 = V(x)$. Now $V'(x) = 2\pi h x - (3\pi h/r)x^2 = \pi h x(2 - 3x/r)$. So $V'(x) = 0 \Rightarrow x = 0$ or $x = \frac{2}{3}r$. The maximum clearly occurs when $x = \frac{2}{3}r$ and then the volume is $\pi h x^2 - (\pi h/r)x^3 = \pi h x^2(1 - x/r) = \pi\left(\frac{2}{3}r\right)^2 h\left(1 - \frac{2}{3}\right) = \frac{4}{27}\pi r^2 h$.

28.



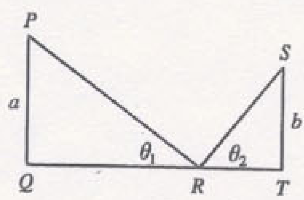
In isosceles triangle AOB, $\angle O = 180° - \theta - \theta$, so $\angle BOC = 2\theta$. The distance rowed is $4\cos\theta$ while the distance walked is the length of arc $BC = 2(2\theta) = 4\theta$. The time taken is given by

$$T(\theta) = \frac{4\cos\theta}{2} + \frac{4\theta}{4} = 2\cos\theta + \theta, \quad 0 \le \theta \le \tfrac{\pi}{2}.$$

$$T'(\theta) = -2\sin\theta + 1 = 0 \Leftrightarrow \sin\theta = \tfrac{1}{2} \Rightarrow \theta = \tfrac{\pi}{6}.$$

Check the value of T at $\theta = \frac{\pi}{6}$ and at the endpoints of the domain of T; that is, $\theta = 0$ and $\theta = \frac{\pi}{2}$. $T(0) = 2$, $T\left(\frac{\pi}{6}\right) = \sqrt{3} + \frac{\pi}{6} \approx 2.26$, and $T\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \approx 1.57$. Therefore, the minimum value of T is $\frac{\pi}{2}$ when $\theta = \frac{\pi}{2}$; that is, the woman should walk all the way. Note that $T''(\theta) = -2\cos\theta < 0$ for $0 \le \theta < \frac{\pi}{2}$, so $\theta = \frac{\pi}{6}$ gives a maximum time.

34.



If $d = |QT|$, we minimize $f(\theta_1) = |PR| + |RS| = a\csc\theta_1 + b\csc\theta_2$. Differentiating with respect to θ_1, and setting $\frac{df}{d\theta_1}$ equal to 0, we get

$$\frac{df}{d\theta_1} = 0 = -a\csc\theta_1\cot\theta_1 - b\csc\theta_2\cot\theta_2\frac{d\theta_2}{d\theta_1}.$$

So we need to find an expression for $\frac{d\theta_2}{d\theta_1}$. We can do this by observing that $|QT| = \text{constant} = a\cot\theta_1 + b\cot\theta_2$.

Differentiating this equation implicitly with respect to θ_1, we get $-a\csc^2\theta_1 - b\csc^2\theta_2\frac{d\theta_2}{d\theta_1} = 0 \Rightarrow$

$\frac{d\theta_2}{d\theta_1} = -\frac{a\csc^2\theta_1}{b\csc^2\theta_2}$. We substitute this into the expression for $\frac{df}{d\theta_1}$ to get

$$-a\csc\theta_1\cot\theta_1 - b\csc\theta_2\cot\theta_2\left(-\frac{a\csc^2\theta_1}{b\csc^2\theta_2}\right) = 0 \Leftrightarrow -a\csc\theta_1\cot\theta_1 + a\frac{\csc^2\theta_1\cot\theta_2}{\csc\theta_2} = 0 \Leftrightarrow$$

$\cot\theta_1\csc\theta_2 = \csc\theta_1\cot\theta_2 \Leftrightarrow \frac{\cot\theta_1}{\csc\theta_1} = \frac{\cot\theta_2}{\csc\theta_2} \Leftrightarrow \cos\theta_1 = \cos\theta_2$. Since θ_1 and θ_2 are both acute, we have $\theta_1 = \theta_2$.

51. $\lim_{x\to\infty} \frac{e^x}{x^n} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}} \overset{H}{=} \cdots \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{n!} = \infty$

52. $\lim_{x\to\infty} \frac{\ln x}{x^p} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0$ since $p > 0$.

54. (a) $\lim_{t\to\infty} v = \lim_{t\to\infty} \frac{mg}{c}\left(1 - e^{-ct/m}\right) = \frac{mg}{c} \lim_{t\to\infty}\left(1 - e^{-ct/m}\right)$

$= \frac{mg}{c}(1-0)$ [because $-ct/m \to -\infty$ as $t \to \infty$] $= \frac{mg}{c}$,

which is the speed the object approaches as time goes on, the so-called limiting velocity.

(b) $\lim_{m\to\infty} v = \lim_{m\to\infty} \frac{mg}{c}\left(1 - e^{-ct/m}\right) = \frac{g}{c} \lim_{m\to\infty} \frac{1 - e^{-ct/m}}{1/m} \overset{H}{=} \frac{g}{c} \lim_{m\to\infty} \frac{-e^{-ct/m}\left(ct/m^2\right)}{-1/m^2}$

$= \frac{g}{c}(ct) \lim_{m\to\infty} e^{-ct/m} = gt(1)$ [because $-ct/m \to 0$ as $m \to \infty$] $= gt$.

The speed of a very heavy falling object is approximately proportional to the elapsed time—it doesn't depend on the mass.