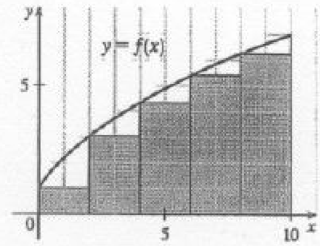
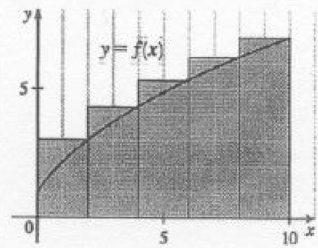
Math 1a Homework Solutions
Section 5.1

1. (a) Since f is *increasing*, we can obtain a *lower* estimate by using *left* endpoints. We are instructed to use five rectangles, so $n = 5$.



$$L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x \quad [\Delta x = \tfrac{b-a}{n} = \tfrac{10-0}{5} = 2]$$
$$= f(x_0)\cdot 2 + f(x_1)\cdot 2 + f(x_2)\cdot 2 + f(x_3)\cdot 2 + f(x_4)\cdot 2$$
$$= 2\,[f(0) + f(2) + f(4) + f(6) + f(8)]$$
$$\approx 2(1 + 3 + 4.3 + 5.4 + 6.3) = 2(20) = 40$$

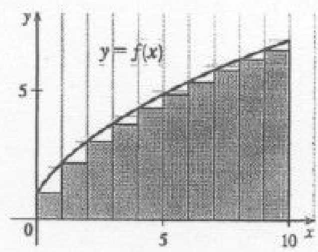
Since f is *increasing*, we can obtain an *upper* estimate by using *right* endpoints.



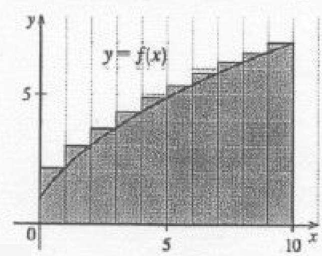
$$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x$$
$$= 2\,[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$$
$$= 2\,[f(2) + f(4) + f(6) + f(8) + f(10)]$$
$$\approx 2(3 + 4.3 + 5.4 + 6.3 + 7) = 2(26) = 52$$

Comparing R_5 to L_5, we see that we have added the area of the rightmost rectangle, $f(10)\cdot 2$, to the sum and subtracted the area of the leftmost rectangle, $f(0)\cdot 2$, from the sum.

(b) $L_{10} = \sum_{i=1}^{10} f(x_{i-1})\,\Delta x \quad [\Delta x = \tfrac{10-0}{10} = 1]$

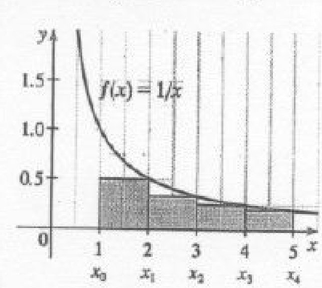


$$= 1\,[f(x_0) + f(x_1) + \cdots + f(x_9)]$$
$$= f(0) + f(1) + \cdots + f(9)$$
$$\approx 1 + 2.1 + 3 + 3.7 + 4.3 + 4.9 + 5.4 + 5.8 + 6.3 + 6.7$$
$$= 43.2$$



$$R_{10} = \sum_{i=1}^{10} f(x_i)\,\Delta x = f(1) + f(2) + \cdots + f(10)$$
$$= L_{10} + 1\cdot f(10) - 1\cdot f(0) \quad \begin{bmatrix}\text{add rightmost rectangle,}\\ \text{subtract leftmost}\end{bmatrix}$$
$$= 43.2 + 7 - 1 = 49.2$$

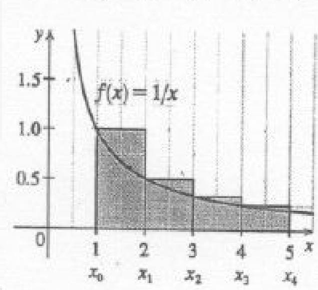
3. (a) $R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x \quad [\Delta x = \tfrac{5-1}{4} = 1]$



$$= f(x_1)\cdot 1 + f(x_2)\cdot 1 + f(x_3)\cdot 1 + f(x_4)\cdot 1$$
$$= f(2) + f(3) + f(4) + f(5)$$
$$= \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} = \tfrac{77}{60} = 1.28\overline{3}$$

Since f is *decreasing* on $[1, 5]$, an *underestimate* is obtained by using the *right* endpoint approximation, R_4.

(b) $L_4 = \sum_{i=1}^{4} f(x_{i-1})\,\Delta x$



$$= f(1) + f(2) + f(3) + f(4)$$
$$= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} = \tfrac{25}{12} = 2.08\overline{3}$$

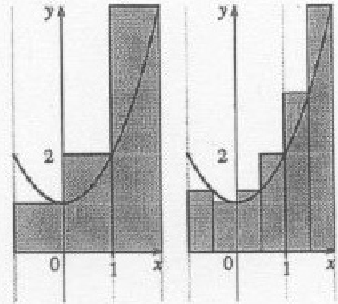
L_4 is an overestimate. Alternatively, we could just add the area of the leftmost rectangle and subtract the area of the rightmost; that is, $L_4 = R_4 + f(1)\cdot 1 - f(5)\cdot 1$.

5. (a) $f(x) = 1 + x^2$ and $\Delta x = \dfrac{2-(-1)}{3} = 1 \quad \Rightarrow$

$$R_3 = 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2) = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 5 = 8.$$

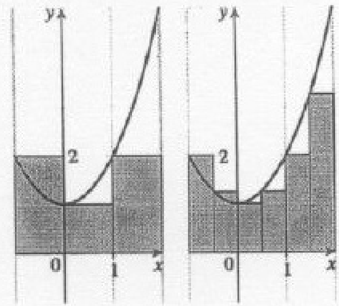
$$\Delta x = \frac{2-(-1)}{6} = 0.5 \quad \Rightarrow$$

$$\begin{aligned} R_6 &= 0.5[f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)] \\ &= 0.5(1.25 + 1 + 1.25 + 2 + 3.25 + 5) \\ &= 0.5(13.75) = 6.875 \end{aligned}$$



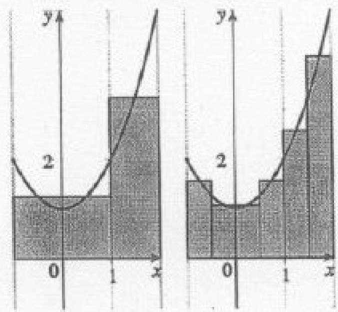
(b) $L_3 = 1 \cdot f(-1) + 1 \cdot f(0) + 1 \cdot f(1) = 1 \cdot 2 + 1 \cdot 1 + 1 \cdot 2 = 5$

$$\begin{aligned} L_6 &= 0.5[f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5)] \\ &= 0.5(2 + 1.25 + 1 + 1.25 + 2 + 3.25) \\ &= 0.5(10.75) = 5.375 \end{aligned}$$



(c)
$$\begin{aligned} M_3 &= 1 \cdot f(-0.5) + 1 \cdot f(0.5) + 1 \cdot f(1.5) \\ &= 1 \cdot 1.25 + 1 \cdot 1.25 + 1 \cdot 3.25 = 5.75 \end{aligned}$$

$$\begin{aligned} M_6 &= 0.5[f(-0.75) + f(-0.25) + f(0.25) \\ &\qquad + f(0.75) + f(1.25) + f(1.75)] \\ &= 0.5(1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625) \\ &= 0.5(11.875) = 5.9375 \end{aligned}$$



(d) M_6 appears to be the best estimate.

7. Here is one possible algorithm (ordered sequence of operations) for calculating the sums:

1 Let SUM $= 0$, X_MIN $= 0$, X_MAX $= \pi$, N $= 10$ (or 30 or 50, depending on which sum we are calculating), DELTA_X = (X_MAX − X_MIN)/N, and RIGHT_ENDPOINT = X_MIN + DELTA_X.

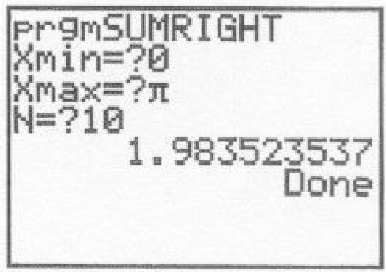
2 Repeat steps 2a, 2b in sequence until RIGHT_ENDPOINT > X_MAX.

2a Add sin (RIGHT_ENDPOINT) to SUM.

2b Add DELTA_X to RIGHT_ENDPOINT.

At the end of this procedure, (DELTA_X) · (SUM) is equal to the answer we are looking for. We find that $R_{10} = \frac{\pi}{10}\sum_{i=1}^{10}\sin\left(\frac{i\pi}{10}\right) \approx 1.9835$, $R_{30} = \frac{\pi}{30}\sum_{i=1}^{30}\sin\left(\frac{i\pi}{30}\right) \approx 1.9982$, and $R_{50} = \frac{\pi}{50}\sum_{i=1}^{50}\sin\left(\frac{i\pi}{50}\right) \approx 1.9993$. It appears that the exact area is 2.

Shown below is program SUMRIGHT and its output from a TI-83 Plus calculator. To generalize the program, we have input (rather than assign) values for Xmin, Xmax, and N. Also, the function, $\sin x$, is assigned to Y_1, enabling us to evaluate any right sum merely by changing Y_1 and running the program.



8. We can use the algorithm from Exercise 7 with X_MIN $= 1$, X_MAX $= 2$, and $1/(\text{RIGHT\_ENDPOINT})^2$ instead of sin (RIGHT_ENDPOINT) in step 2a. We find that $R_{10} = \frac{1}{10}\sum_{i=1}^{10}\frac{1}{(1+i/10)^2} \approx 0.4640$, $R_{30} = \frac{1}{30}\sum_{i=1}^{30}\frac{1}{(1+i/30)^2} \approx 0.4877$, and $R_{50} = \frac{1}{50}\sum_{i=1}^{50}\frac{1}{(1+i/50)^2} \approx 0.4926$. It appears that the exact area is $\frac{1}{2}$.