Math 1A Fall 2001: Section 3.6 Solutions

2. (a) $\frac{d}{dx}\left(4x^2+9y^2\right)=\frac{d}{dx}(36) \Rightarrow 8x+18y\cdot y'=0 \Rightarrow y'=-\frac{8x}{18y}=-\frac{4x}{9y}$

(b) $4x^2+9y^2=36 \Rightarrow 9y^2=36-4x^2 \Rightarrow y^2=\frac{4}{9}(9-x^2) \Rightarrow y=\pm\frac{2}{3}\sqrt{9-x^2}$, so $y'=\pm\frac{2}{3}\cdot\frac{1}{2}(9-x^2)^{-1/2}(-2x)=\mp\frac{2x}{3\sqrt{9-x^2}}$

(c) From part (a), $y'=-\frac{4x}{9y}=-\frac{4x}{9\left(\pm\frac{2}{3}\sqrt{9-x^2}\right)}=\mp\frac{2x}{3\sqrt{9-x^2}}$.

4. $\frac{d}{dx}\left(x^2-2xy+y^3\right)=\frac{d}{dx}(c) \Rightarrow 2x-2(xy'+y\cdot 1)+3y^2y'=0 \Rightarrow 2x-2y=2xy'-3y^2y' \Rightarrow$ $2x-2y=y'(2x-3y^2) \Rightarrow y'=\frac{2x-2y}{2x-3y^2}$

8. $\sqrt{1+x^2y^2}=2xy \Rightarrow \frac{1}{2}(1+x^2y^2)^{-1/2}(x^2\cdot 2yy'+y^2\cdot 2x)=2(xy'+y\cdot 1) \Rightarrow$

$$\frac{2x^2y}{2\sqrt{1+x^2y^2}}y'+\frac{2xy^2}{2\sqrt{1+x^2y^2}}=2xy'+2y \Rightarrow y'\left(\frac{x^2y}{\sqrt{1+x^2y^2}}-2x\right)=2y-\frac{xy^2}{\sqrt{1+x^2y^2}} \Rightarrow$$

$$y'\left(\frac{x^2y-2x\sqrt{1+x^2y^2}}{\sqrt{1+x^2y^2}}\right)=\frac{2y\sqrt{1+x^2y^2}-xy^2}{\sqrt{1+x^2y^2}} \Rightarrow$$

$$y'=\frac{2y\sqrt{1+x^2y^2}-xy^2}{x^2y-2x\sqrt{1+x^2y^2}}=\frac{y\left(2\sqrt{1+x^2y^2}-xy\right)}{x\left(xy-2\sqrt{1+x^2y^2}\right)}=-\frac{y}{x}$$

12. $\sin x+\cos y=\sin x\cos y \Rightarrow \cos x-\sin y\cdot y'=\sin x(-\sin y\cdot y')+\cos y\cos x \Rightarrow$ $(\sin x\sin y-\sin y)y'=\cos x\cos y-\cos x \Rightarrow y'=\frac{\cos x(\cos y-1)}{\sin y(\sin x-1)}$

16. $x^{2/3}+y^{2/3}=4 \Rightarrow \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0 \Rightarrow \frac{1}{\sqrt[3]{x}}+\frac{y'}{\sqrt[3]{y}}=0 \Rightarrow y'=-\frac{\sqrt[3]{y}}{\sqrt[3]{x}}$. When $x=-3\sqrt{3}$ and $y=1$, we have $y'=-\frac{1}{(-3\sqrt{3})^{1/3}}=-\frac{(-3\sqrt{3})^{2/3}}{-3\sqrt{3}}=\frac{3}{3\sqrt{3}}=\frac{1}{\sqrt{3}}$, so an equation of the tangent line is $y-1=\frac{1}{\sqrt{3}}(x+3\sqrt{3})$ or $y=\frac{1}{\sqrt{3}}x+4$.

26. $x^2+6xy+y^2=8 \Rightarrow 2x+6xy'+6y+2yy'=0 \Rightarrow (3x+y)y'=-x-3y \Rightarrow$

$$y'=-\frac{x+3y}{3x+y} \Rightarrow$$

$$y''=-\frac{(3x+y)(1+3y')-(x+3y)(3+y')}{(3x+y)^2}=-\frac{-8y+8xy'}{(3x+y)^2}=\frac{8(y-xy')}{(3x+y)^2}$$

$$=\frac{8[y-x(-x-3y)/(3x+y)]}{(3x+y)^2}\cdot\frac{3x+y}{3x+y}=\frac{8[y(3x+y)+x(x+3y)]}{(3x+y)^3}$$

$$=\frac{8(x^2+6xy+y^2)}{(3x+y)^3}=\frac{64}{(3x+y)^3}$$

At the last step, we used the fact that x and y must satisfy the original equation, $x^2+6xy+y^2=8$.

34. Let $y=\cos^{-1}x$. Then $\cos y=x$ and $0\le y\le\pi \Rightarrow \frac{d}{dx}(\cos y)=\frac{d}{dx}(x) \Rightarrow -\sin y\frac{dy}{dx}=1 \Rightarrow$ $\frac{dy}{dx}=-\frac{1}{\sin y}=-\frac{1}{\sqrt{1-\cos^2 y}}=-\frac{1}{\sqrt{1-x^2}}$ (Note that $\sin y\ge 0$ for $0\le y\le\pi$.)

38. $x^2-y^2=5$ and $4x^2+9y^2=72$ intersect when $4x^2+9(x^2-5)=72 \Leftrightarrow 13x^2=117 \Leftrightarrow x=\pm 3$, so there are four points of intersection: $(\pm 3,\pm 2)$. $x^2-y^2=5 \Rightarrow 2x-2yy'=0 \Rightarrow y'=x/y$, and $4x^2+9y^2=72 \Rightarrow 8x+18yy'=0 \Leftrightarrow y'=-4x/9y$. At $(3,2)$, the slopes are $m_1=\frac{3}{2}$ and $m_2=-\frac{2}{3}$, so the curves are orthogonal. By symmetry, the curves are also orthogonal at $(3,-2)$, $(-3,2)$ and $(-3,-2)$.

54. $x^2+4y^2=5 \Rightarrow 2x+4(2yy')=0 \Rightarrow y'=-\frac{x}{4y}$. Now let h be the height of the lamp, and let (a,b) be the point of tangency of the line passing through the points $(3,h)$ and $(-5,0)$. This line has slope $(h-0)/[3-(-5)]=\frac{1}{8}h$. But the slope of the tangent line through the point (a,b) can be expressed as $y'=-\frac{a}{4b}$, or as $\frac{b-0}{a-(-5)}=\frac{b}{a+5}$ [since the line passes through $(-5,0)$ and (a,b)], so $-\frac{a}{4b}=\frac{b}{a+5} \Leftrightarrow$ $4b^2=-a^2-5a \Leftrightarrow a^2+4b^2=-5a$. But $a^2+4b^2=5$ [since (a,b) is on the ellipse], so $5=-5a \Leftrightarrow$ $a=-1$. Then $4b^2=-a^2-5a=-1-5(-1)=4 \Rightarrow b=1$, since the point is on the top half of the ellipse. So $\frac{h}{8}=\frac{b}{a+5}=\frac{1}{-1+5}=\frac{1}{4} \Rightarrow h=2$. So the lamp is located 2 units above the x-axis.